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CONVERGENCE RATES OF FINITE DIFFERENCE SCHEMES FOR THE WAVE EQUATION WITH ROUGH COEFFICIENTS

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ABSTRACT. The propagation of acoustic waves in a rough heterogeneous medium is modeled using the linear wave equation with a variable but merely Hölder continuous coefficient. We design robust finite difference discretizations that are shown to converge to the weak solution. We rigorously determine the rate of convergence of these discretizations by an L^2 variant of the Kruzkhov doubling of variables technique. Numerical experiments illustrating these rates of convergence are also presented.

1. INTRODUCTION

Propagation of acoustic waves in a heterogeneous medium plays a significant role in many applications, for instance in seismic imaging in geophysics and in the exploration of hydrocarbons [1, 7]. This wave propagation is modeled by the linear wave equation:

$$p_{tt}(t,\mathbf{x}) - \operatorname{div}(c(\mathbf{x})\nabla p(t,\mathbf{x})) = 0, \quad (t,\mathbf{x}) \in D_T, \tag{1.1a}$$

$$p(0,\mathbf{x}) = p_0(\mathbf{x}), \quad \mathbf{x} \in D, \tag{1.1b}$$

$$p_t(0,\mathbf{x}) = p_1(\mathbf{x}), \quad \mathbf{x} \in D, \tag{1.1c}$$

where $D_T := [0,T] \times D$, $D \subset \mathbb{R}^d$, augmented with periodic or homogeneous Dirichlet boundary conditions (and the functions extended by zero outside of the domain). Here, p is the acoustic pressure and the wave speed is determined by the coefficient $c = c(\mathbf{x}) > 0$. The coefficient c encodes information about the material properties of the medium. As an example, the coefficient c could represent rock permeability when seismic waves propagate in a rock formation.

It is well known that the linear wave equation (1.1) can be rewritten as a first-order system of partial differential equations by $u(t,x) := p_t(t,x)$ and $\mathbf{r}(t,\mathbf{x}) := \nabla p(t,\mathbf{x})$, resulting in

$$\begin{aligned} u_t(t,\mathbf{x}) - \operatorname{div}(c(\mathbf{x})\mathbf{r}(t,\mathbf{x})) &= 0, \\ \mathbf{r}_t(t,\mathbf{x}) - \nabla u(t,\mathbf{x}) &= 0, \quad (t,\mathbf{x}) \in D_T, \end{aligned} \tag{1.2a}$$

$$u(0,\mathbf{x}) = p_1(\mathbf{x}), \quad \mathbf{x} \in D, \tag{1.2b}$$

$$\mathbf{r}(0,\mathbf{x}) = \nabla p_0(\mathbf{x}), \quad \mathbf{x} \in D. \tag{1.2c}$$

The above system (1.2) is strictly hyperbolic [5] with wave speeds given by $\pm\sqrt{c}$. Under the assumption that the coefficient $c \in C^{0,\alpha} \cap L^\infty(D)$ for some $\alpha > 0$ and that it is uniformly positive on D i,e there exists constants $\underline{c}, \overline{c} > 0$ such that

$$0 < \underline{c} \le c(\mathbf{x}) \le \overline{c}, \quad \forall \mathbf{x} \in D. \tag{1.3}$$

and that the initial data $p_0 \in H^1(D)$ and $p_1 \in L^2(D)$, one can prove existence of a unique weak solution $p \in C^0([0,T];H^1(D))$ with $p_t \in C^0([0,T];L^2(D))$ following classical energy arguments for linear partial differential equations. See for instance [11, Chapter III, Theorems 8.1 and 8.2]. A smoother coefficient c and more regular initial data p_0, p_1 result in a more regular solution [11].

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1.1. **Numerical schemes for the wave equation.** Although the wave equation (1.1) is linear, the presence of a material coefficient c and (possibly) complex geometry of the domain D imply that that analytical solution formulas for (1.1) are not available. Consequently, numerical approximation plays a very significant role in the modeling of acoustic wave propagation in heterogeneous media with complex domain geometry.

A popular class of methods for discretizing the wave equation are the finite difference methods [5, 8]. Within this framework, the equivalent first-order hyperbolic system (1.2) is discretized on a grid with the spatial differential operators being replaced by central finite differences (of the appropriate order). Temporal discretization is typically performed using high-order Runge-Kutta methods. Finite difference approximations are simple and efficient, particularly on Cartesian or (block) structured grids. Convergence analysis for these methods for the initial-boundary value problem for the wave equation is fairly classical, see [5].

Another popular class of methods for discretizing the wave equation are of the finite element type [10]. In this framework, a variational formulation of the second-order version (1.1) of the wave equation is discretized using suitable (polynomial) finite element subspaces. Convergence analysis for the finite element method is presented in [10] and references therein. Other methods such as boundary element methods and spectral methods are not commonly used for discretizing (1.1) on account of the heterogeneity in the coefficient.

A key question in numerical analysis for partial differential equations is *the rate at which the approximate solutions (generated by the discretizations) converge to the exact solution of the equation*. There is considerable literature on the convergence rates for both the finite difference and finite element discretizations, see [5] and [10]. For both methods, the essential result for convergence rates can be expressed heuristically as

If the exact solution is smooth enough, then the finite difference discretization converges at the rate of the truncation error (determined by the order of the spatial and temporal discretization) and the finite element scheme converges at the rate of the underlying polynomial approximation

Hence, the key issue in obtaining the correct convergence rate for a given numerical method is the regularity of the solution of the underlying PDE. If the coefficient c and the initial data p_0, p_1 are smooth, say $C^k(D)$ or $H^s(D)$ for some large enough Sobolev exponent s, then by regularity results for the linear wave equation [11], the solution also is smooth i.e, it belongs to $H^s(D_T)$ and the finite difference (resp. finite element) discretizations converge at the order of the underlying difference operators (resp. polynomial approximation spaces).

1.2. **Rough coefficients.** As noted above, the regularity of the solution to the wave equation (1.1) and the resulting (high) rate of convergence of numerical approximations relies on the smoothness of the coefficient c. Consequently, most of the numerical analysis literature on the wave equation assumes a smooth coefficient c. However, this assumption is not realized in practice. As noted before, the wave equation is heavily used to model seismic imaging in rock formations and other porous media (for instance oil and gas reservoirs). Such media are very heterogenous with sharp interfaces, strong contrasts and aspect ratios [7]. Furthermore, the material properties of such media can only be determined by measurements. Such measurements are inherently *uncertain*. This uncertainty is modeled in a statistical manner by representing the material properties (such as rock permeability) as random fields. In particular, log-normal random fields are heavily used in modeling material properties in porous and other geophysically relevant media [7, 4]. Consequently, the coefficient c is not smooth, not even continuously differentiable, see figure 1 for an illustration of coefficient c whereas the rock permeability is modeled by a log-normal random field (the figure represents a single realization of the field). Closer inspection of the coefficients obtained in practice reveals that at most, the material coefficient c is a Hölder continuous function i.e, $c \in C^{0,\alpha}$ for some $0 < \alpha < 1$. No further regularity can be assumed on the coefficient c represent material properties of most geophysical formations.

Given the above discussion, it is natural to search for numerical methods that can effectively and efficiently approximate the acoustic wave equation with rough (merely Hölder continuous)

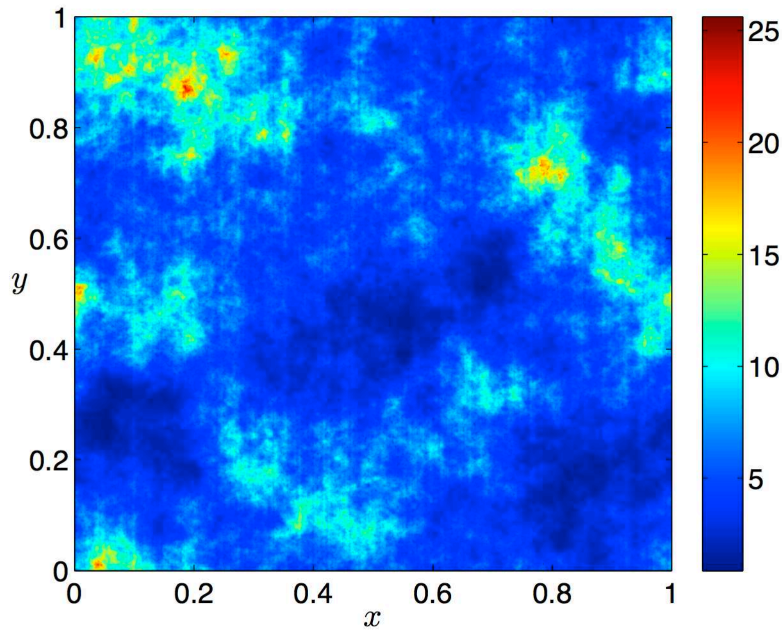

FIGURE 1. The coefficient c in the wave equation (1.1) in two dimensions as a single realization of a log-normal random field

coefficients. In particular, one is interested in designing numerical methods that can be rigorously shown to converge to the underlying weak solution (note that the weak solution exists and is unique even when the coefficient is merely Hölder continuous). Furthermore, one is also interested in obtaining (rigorously) a convergence rate for the discretization as the mesh parameters are refined. We remark that the issue of a convergence rate is not just of theoretical significance, it has profound implications on calculating complexity estimates for Monte-Carlo and Multi-level Monte Carlo methods (see [12, 13, 17]) to solve the random (uncertain) PDE that results from considering the material coefficient as a random field (as is done in engineering practice).

An extensive search through the literature revealed that no rigorous numerical analysis results are available for the case of convergence rates for numerical approximations to the wave equation with rough material coefficients. All available convergence rate results (for both finite difference as well as finite element approximations) strictly require a smooth (at least C^1 material coefficient). Given this paucity of available results, we consider this issue in the current paper.

1.3. **Aims and scope of the current paper.** The central aims of the current paper are as follows,

- To design a numerical scheme for approximating the acoustic wave equation with a rough, merely Hölder continuous, material coefficient and to show that this scheme converges to the weak solution of the underlying PDE.
- To obtain (rigorously) a rate of convergence for this scheme to the exact solution as the mesh parameters are refined.

To this end, we construct suitable fully discrete *upwind* finite difference discretizations of the wave equation with a rough coefficient, represented by the first-order hyperbolic system (1.2). Given the low regularity of the coefficient, also inherited by the solution, the solution is expected to have possibly sharp interfaces and contrasts making upwinding necessary for numerical stability. Next, we obtain energy estimates for the approximate solution and use them to prove convergence to a weak solution as the mesh is refined.

The key part of our paper is the determination of a convergence rate for the numerical approximation. Convergence rates for standard finite difference approximations use the truncation error technique [5] and require that the underlying solution be smooth enough. Similarly, convergence rates for the finite element approximation use best approximation rates for the underlying polynomial spaces and again need regularity. Given the lack of regularity for our solution (note that $p \in H^1$), this technique is not adequate for our purposes. Hence, we needed to find an innovative approach for determining the rate of convergence.

Motivated by the convergence theory for numerical approximations of scalar conservation laws due to Kuznetsov (see [6] and references therein), wherein the Kruzkhov doubling of variables technique [9] is adapted to compare a numerical solution with an exact solution with respect to the L^1 norm in space and a suitable rate of convergence is obtained, we modify this approach in our L^2 (energy space) setting. We define a *novel* doubling of variables technique in L^2 and use it to obtain a rate of convergence for the finite difference approximation. The resulting rate is dependent on the Hölder coefficient α of the coefficient c as well as on the modulus of continuity in L^2 that measures the regularity of the initial data. In particular, we obtain that a rougher coefficient yields a slower convergence rate, consistent with empirical observations [17]. Numerical examples illustrating this phenomena as well as investigating the optimality of the obtained rates are also presented. To the best of our knowledge, our results are the first rigorous rate of convergence results for numerical approximation to the wave equation with rough coefficients.

The rest of the paper in organized as follows: in section 2, we consider the one-dimensional version of the wave equation (1.1) and prove convergence rates for a finite difference scheme. The two-dimensional version is considered in section 3 and the contents of the paper are summarized in section 4.

2. The one-dimensional case

For simplicity of exposition as well as to illustrate the techniques, we start with the acoustic wave equation (1.2) in one space dimension:

$$\begin{aligned} u_t(t,x) - (c(x)r(t,x))_x &= 0, \\ r_t(t,x) - u_x(t,x) &= 0, \quad (t,x) \in D_T, \end{aligned} \tag{2.1}$$

$D = [d_L, d_R]$, $d_L < d_R \in [-\infty, \infty]$.

We will work with an equivalent system that results from (2.1) by defining the variable, $v(t,x) := c(x)r(t,x)$:

$$\begin{aligned} u_t(t,x) - v(t,x)_x &= 0, \\ v_t(t,x) - c(x)u_x(t,x) &= 0, \quad (t,x) \in D_T, \end{aligned} \tag{2.2}$$

2.1. Numerical approximation of (2.2) by a finite difference scheme.

In order to compute numerical approximations to (2.2), we choose $\Delta x > 0$ and discretize the spatial domain by a grid with gridpoints $x_{j+1/2} := j\Delta x$, $j \in \mathbb{Z}$. Similarly let Δt denote the time step and $t^n = n\Delta t$ with $n = 0, 1, \cdots, N$ denote the n-th time level with $N\Delta t = T$.

We define the averaged quantities

$$c_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} c(x)\,dx, \quad j \in \mathbb{Z}, \tag{2.3}$$

and

$$(u_j^0, v_j^0) = \frac{1}{\Delta x}\left(\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,dx, \int_{x_{j-1/2}}^{x_{j+1/2}} v_0(x)dx\right) \quad j \in \mathbb{Z}, \tag{2.4}$$

and finally set $r_j^0 := c_j^{-1}v_j^0$. Moreover, we denote, for a quantity σ_j^n, $j \in \mathbb{Z}$, $n = 0, \dots, N_T$ defined on the grid,

$$D_t^+\sigma_j^n := \frac{1}{\Delta t}(\sigma_j^{n+1} - \sigma_j^n), \quad D_x^\pm\sigma_j^n = \pm\frac{1}{\Delta x}(\sigma_{j\pm1}^n - \sigma_j^n), \quad D_x^c\sigma_j^n = \frac{1}{2\Delta x}(\sigma_{j+1}^n - \sigma_{j-1}^n). \tag{2.5}$$

Then we define approximations to (2.2) by the finite difference scheme:

$$D_t^+ u_j^n = D_x^c v_j^n + \frac{\Delta x}{2} D_x^+ D_x^- u_j^n, \tag{2.6a}$$

$$\frac{D_t^+ v_j^n}{c_j} = D_x^c u_j^n + \frac{\Delta x}{2} D_x^+ D_x^- v_j^n, \quad j \in \mathbb{Z}, n = 1, \dots, N, \tag{2.6b}$$

with the time step Δt being chosen such that the CFL-condition,

$$2\Delta t \max_j \{\max\{2c_j + 1, c_j/4 + 5/4\}\} \leq \Delta x \tag{2.7}$$

is satisfied.

Moreover for any $k, l \in \mathbb{R}$, we define the discrete entropy (energy) function and flux

$$\eta_j^n := \frac{|u_j^n - k|^2}{2} + \frac{|v_j^n - \ell|^2}{2c_j}, \quad q_j^n := -(u_j^n - k)(v_j^n - \ell). \tag{2.8}$$

The scheme (2.6) satisfies the following properties:

Lemma 2.1. *Assume $c \in C^{0,\alpha}(D)$ and $u_0, v_0 \in L^2(D)$. Then the numerical approximations u_j^n and v_j^n defined by (2.6), (2.3) and (2.4) have the following properties:*

(i) *Discrete entropy inequality:*

$$D_t^+ \eta_j^n + D_x^c q_j^n \leq \frac{\Delta x\,(\Delta t - \Delta x)}{2} D_x^- \left(D_x^+ \left(u_j^n - k\right) D_x^+ \left(v_j^n - l\right)\right) + \frac{\Delta x}{4} D_x^+ D_x^- \left(\left(u_j^n - k\right)^2 + \left(v_j^n - l\right)^2\right). \tag{2.9}$$

(ii) *Bounds on the discrete L^2-norms:*

$$\Delta x \sum_j (u_j^n)^2 + \frac{1}{c_j}(v_j^n)^2 \leq \Delta x \sum_j (u_j^0)^2 + \frac{1}{c_j}(v_j^0)^2 \leq \|u_0\|_{L^2}^2 + \left\|c^{-1/2} v_0\right\|_{L^2}^2 \tag{2.10}$$

(iii) *For any function $w = w(x)$, define the L^2 modulus of continuity in space as γ if,*

$$\nu_x^2(w, \sigma) := \sup_{\delta \leq \sigma} \int_{\mathbb{R}} |w(x + \delta) - w(x)|^2 \, dx \leq C\, \sigma^{2\gamma}. \tag{2.11}$$

If we also assume that the initial data u_0 and v_0 have moduli of continuity in $L^2(D)$,

$$\nu_x^2(u_0, \sigma) \leq C\sigma^{2\gamma}, \quad \nu_x^2(v_0, \sigma) \leq C\,\sigma^{2\gamma},$$

for some $\gamma > 0$, the approximations satisfy,

$$\begin{aligned} &\Delta x \sum_j \left|D_{\gamma,t}^+ u_j^n\right|^2 + \frac{1}{c_j}\left|D_{\gamma,t}^+ v_j^n\right|^2 \leq C, \\ &\Delta x \sum_j \left|D_{\gamma,x}^c u_j^n\right|^2 + \left|D_{\gamma,x}^c v_j^n\right|^2 + \frac{\Delta x^2}{4}\left(\left|D_{\gamma,x}^+ D_x^- u_j^n\right|^2 + \left|D_{\gamma,x}^+ D_x^- v_j^n\right|^2\right) \leq C, \end{aligned} \tag{2.12}$$

for all $n = 0, \dots, N_T$, where C is a constant depending on c and the initial data u_0 and v_0.

Proof. By linearity, it is sufficient to prove (2.9) for $k = l = 0$. We shall use the following identities

$$u_j^n D_t^+ u_j^n = \frac{1}{2} D_t^+ \left(u_j^n\right)^2 - \frac{\Delta t}{2}\left(D_t^+ u_j^n\right)^2, \tag{2.13}$$

$$u_j^n D_x^+ D_x^- u_j^n = \frac{1}{2} D_x^+ D_x^- \left(u_j^n\right)^2 - \frac{1}{2}\left(\left(D_x^- u_j^n\right)^2 + \left(D_x^+ u_j^n\right)^2\right), \tag{2.14}$$

$$D_x^- \left(D_x^+ u_j^n D_x^+ v_j^n\right) = \left(D_x^+ D_x^- u_j^n\right) D_x^c v_j^n + \left(D_x^+ D_x^- v_j^n\right) D_x^c u_j^n, \tag{2.15}$$

$$u_j^n D_x^c v_j^n + v_j^n D_x^c u_j^n = D_x^c \left(u_j^n v_j^n\right) - \frac{\Delta x^2}{2} D_x^- \left(D_x^+ u_j^n D_x^+ v_j^n\right). \tag{2.16}$$

Multiplying (2.6a) by u_j^n and (2.6b) by v_j^n we get

$$\begin{aligned}\frac{1}{2}D_t^+\left(u_j^n\right)^2 - \frac{\Delta t}{2}\left(D_t^+u_j^n\right)^2 &= u_j^n D_x^c v_j^n + \frac{\Delta x}{4}D_x^+D_x^-\left(u_j^n\right)^2\\ &\quad - \frac{\Delta x}{4}\left(\left(D_x^-u_j^n\right)^2 + \left(D_x^+u_j^n\right)^2\right)\\ \frac{1}{2c_j}D_t^+\left(v_j^n\right)^2 - \frac{\Delta t}{2c_j}\left(D_t^+v_j^n\right)^2 &= v_j^n D_x^c u_j^n + \frac{\Delta x}{4}D_x^+D_x^-\left(v_j^n\right)^2\\ &\quad - \frac{\Delta x}{4}\left(\left(D_x^-v_j^n\right)^2 + \left(D_x^+v_j^n\right)^2\right).\end{aligned}$$

Adding these two equations

$$\begin{aligned}D_t^+\eta_j^n &= D_x^c\left(u_j^n v_j^n\right) - \frac{\Delta x^2}{2}D_x^-\left(D_x^+u_j^n D_x^+v_j^n\right)\\ &\quad + \frac{\Delta x}{4}D_x^+D_x^-\left(\left(u_j^n\right)^2 + \left(v_j^n\right)^2\right)\\ &\quad - \frac{\Delta x}{4}\left(\left(D_x^-u_j^n\right)^2 + \left(D_x^+u_j^n\right)^2 + \left(D_x^-v_j^n\right)^2 + \left(D_x^+v_j^n\right)^2\right)\\ &\quad + \frac{\Delta t}{2}\Bigg[\underbrace{\left(D_x^c v_j^n + \frac{\Delta x}{2}D_x^+D_x^-u_j^n\right)^2 + c_j\left(D_x^c u_j^n + \frac{\Delta x}{2}D_x^-D_x^+v_j^n\right)^2}_{a}\Bigg].\end{aligned}$$

We can estimate a as follows

$$\begin{aligned}a &\le \frac{1}{2}\left(\left(D_x^-u_j^n\right)^2 + \left(D_x^+u_j^n\right)^2 + c_j\left(D_x^-v_j^n\right)^2 + c_j\left(D_x^+v_j^n\right)^2\right)\\ &\quad + \Delta x\left(D_x^+D_x^-u_j^n D_x^c v_j^n + c_j D_x^+D_x^-v_j^n D_x^c u_j^n\right) + \frac{\Delta x^2}{4}\left(\left(D_x^+D_x^-u_j^n\right)^2 + c_j\left(D_x^+D_x^-v_j^n\right)^2\right)\\ &\le \left(D_x^-u_j^n\right)^2 + \left(D_x^+u_j^n\right)^2 + c_j\left(D_x^-v_j^n\right)^2 + c_j\left(D_x^+v_j^n\right)^2\\ &\quad + \Delta x\left(D_x^+D_x^-u_j^n D_x^c v_j^n + c_j D_x^+D_x^-v_j^n D_x^c u_j^n\right)\\ &= \left(D_x^-u_j^n\right)^2 + \left(D_x^+u_j^n\right)^2 + c_j\left(D_x^-v_j^n\right)^2 + c_j\left(D_x^+v_j^n\right)^2\\ &\quad + \Delta x D_x^+\left(D_x^-u_j^n D_x^-v_j^n\right) + \Delta x\left(c_j - 1\right)D_x^+D_x^-v_j^n D_x^c u_j^n,\\ &\le \left(D_x^-u_j^n\right)^2 + \left(D_x^+u_j^n\right)^2 + c_j\left(D_x^-v_j^n\right)^2 + c_j\left(D_x^+v_j^n\right)^2\\ &\quad + \Delta x D_x^+\left(D_x^-u_j^n D_x^-v_j^n\right)\\ &\quad + \frac{1}{2}\left|c_j - 1\right|\left(\left(\left|D_x^+v_j^n\right| + \left|D_x^-v_j^n\right|\right)^2 + \frac{1}{4}\left(D_x^-u_j^n + D_x^+u_j^n\right)^2\right)\\ &\le \Delta x D_x^+\left(D_x^-u_j^n D_x^-v_j^n\right) + \left(1 + \frac{1}{4}\left|c_j - 1\right|\right)\left(D_x^-u_j^n\right)^2 + \left(1 + \frac{1}{4}\left|c_j - 1\right|\right)\left(D_x^+u_j^n\right)^2\\ &\quad + \left(c_j + \left|c_j - 1\right|\right)\left(D_x^-v_j^n\right)^2 + \left(c_j + \left|c_j - 1\right|\right)\left(D_x^+v_j^n\right)^2.\end{aligned}$$

This implies that

$$\begin{aligned}D_t^+\eta_j^n + D_x^c q_j^n &\le \frac{\Delta x\left(\Delta t - \Delta x\right)}{2}D_x^-\left(D_x^+u_j^n D_x^+v_j^n\right)\\ &\quad + \frac{\Delta x}{4}D_x^+D_x^-\left(\left(u_j^n\right)^2 + \left(v_j^n\right)^2\right)\\ &\qquad + \frac{1}{2}\left(\left(1 + \frac{1}{4}\left|c_j - 1\right|\right)\Delta t - \frac{\Delta x}{2}\right)\left(D_x^-u_j^n\right)^2\\ &\qquad + \frac{1}{2}\left(\left(1 + \frac{1}{4}\left|c_j - 1\right|\right)\Delta t - \frac{\Delta x}{2}\right)\left(D_x^+u_j^n\right)^2\end{aligned}$$

$$+\frac{1}{2}\left(\left(c_j+|c_j-1|\right)\Delta t-\frac{\Delta x}{2}\right)\left(D_x^- v_j^n\right)^2$$
$$+\frac{1}{2}\left(\left(c_j+|c_j-1|\right)\Delta t-\frac{\Delta x}{2}\right)\left(D_x^+ v_j^n\right)^2.$$

If Δt satisfies the CFL-condition (2.7), the four last terms above are non-positive and (2.9) follows. The L^2 bound (2.10) also follows upon summing over j and multiplying by Δx.

By the linearity of the equation, (2.10) also holds for the difference of two approximations computed by (2.6a) and (2.6b), thus in particular for $D^+_{\gamma,t}u_j^n$ and $D^+_{\gamma,t}v_j^n$. Hence, using the handy equality

$$\sum_j \left|D_t^+ u_j^n\right|^2+\frac{1}{c_j^2}\left|D_t^+ v_j^n\right|^2 \tag{2.17}$$
$$=\sum_j \left|D_x^c u_j^n\right|^2+\left|D_x^c v_j^n\right|^2+\frac{\Delta x^2}{4}\left(\left|D_x^+D_x^- u_j^n\right|^2+\left|D_x^+D_x^- v_j^n\right|^2\right),$$

the CFL-condition (2.7), (2.10) implies

$$\Delta x\sum_j \left(D^+_{\gamma,t}u_j^n\right)^2+\frac{1}{c_j}\left(D^+_{\gamma,t}v_j^n\right)^2 \tag{2.18}$$
$$\begin{aligned}
&\le \Delta x\sum_j \left(D^+_{\gamma,t}u_j^0\right)^2+\frac{1}{c_j}\left(D^+_{\gamma,t}v_j^0\right)^2\\
&\le \max\{1,\bar c\}\Delta x\sum_j \left(D^+_{\gamma,t}u_j^0\right)^2+\frac{1}{c_j^2}\left(D^+_{\gamma,t}v_j^0\right)^2\\
&= \max\{1,\bar c\}\Delta x\Delta t^{2-2\gamma}\sum_j \left(D^c_x u_j^0\right)^2+\left(D^c_x v_j^0\right)^2\\
&\qquad\qquad +\frac{\Delta x^2}{4}\left(D_x^+D_x^-u_j^0\right)^2+\left(D_x^+D_x^-v_j^0\right)^2)\\
&\le \max\{1,\bar c\}\Delta x\,\theta^{2-2\gamma}\sum_j \left(D^c_{\gamma,x} u_j^0\right)^2+\left(D^c_{\gamma,x} v_j^0\right)^2\\
&\qquad\qquad +\frac{\Delta x^2}{4}\left(\left(D_{\gamma,x}^+D_x^-u_j^0\right)^2+\left(D_{\gamma,x}^+D_x^-v_j^0\right)^2\right)\\
&\le 2\theta^{2-2\gamma}\max\{1,\bar c\}\Delta x\sum_j \left(D^+_{\gamma,x} u_j^0\right)^2+\left(D^+_{\gamma,x} v_j^0\right)^2 =: C(\alpha,u_0,v_0),
\end{aligned}$$

where we have set $\theta=\Delta t/\Delta x$. Applying (2.17) once more, we also obtain the second equation in (2.12),

$$\Delta x\sum_j \left(D^c_{\gamma,x}u_j^n\right)^2+\left(D^c_{\gamma,x}v_j^n\right)^2+\frac{\Delta x^2}{4}\left(\left(D^+_{\gamma,x}D_x^-u_j^n\right)^2+\left(D^+_{\gamma,x}D_x^-v_j^n\right)^2\right) \tag{2.19}$$
$$=\theta^{2\gamma-2}\Delta x\sum_j \left(D^+_{\gamma,t}u_j^n\right)^2+\frac{1}{c_j^2}\left(D^+_{\gamma,t}v_j^n\right)^2\le C(\alpha,u_0,v_0).$$

□

Defining

$$u_{\Delta x}(t,x)=u_j^n,\quad (t,x)\in[t^n,t^{n+1})\times[x_{j-1/2},x_{j+1/2}), \tag{2.20a}$$
$$v_{\Delta x}(t,x)=v_j^n,\quad (t,x)\in[t^n,t^{n+1})\times[x_{j-1/2},x_{j+1/2}), \tag{2.20b}$$
$$r_{\Delta x}(t,x)=\frac{v_j^n}{c_j},\quad (t,x)\in[t^n,t^{n+1})\times[x_{j-1/2},x_{j+1/2}), \tag{2.20c}$$
$$c_{\Delta x}(x)=c_j,\quad x\in[x_{j-1/2},x_{j+1/2}), \tag{2.20d}$$

we have that by Kolmogorov's compactness theorem, that a subsequence of $(u_{\Delta x}, r_{\Delta x})_{\Delta x>0}$ converges in $C([0,T];L^2(D))$ to a limit $(u,r) \in C([0,T];L^2(D))$ which is a weak solution of (1.2) as $\Delta x \to 0$. Moreover, the couple (u,r) have the same moduli of continuity as the discrete approximations, in particular,

$$u, r \in L^\infty([0,T];H^s(D)) \cap C^{0,\min\{\alpha,\gamma\}}([0,T];L^2(D)) \quad 0 < s \leq \min\{\gamma,\alpha\}. \tag{2.21}$$

The limit is unique, thanks to the linearity of the equation and the entropy inequality. Therefore,

$$\eta(u-k, v-\ell, c)_t + q(u-k, v-\ell)_x \leq 0, \quad \text{in the sense of distributions,} \tag{2.22}$$

where

$$\eta(u,v,c) := \frac{u^2}{2} + \frac{v^2}{2c}, \quad q(u,v) := -uv, \tag{2.23}$$

which follows from (2.10) in the limit $\Delta x \to 0$.

Remark 2.1. *If we assume $u_0 \in H^1(D)$ and $r_0(x) \equiv 0$ in (2.1) (so that $v_0 \equiv 0$), we obtain in the same way that $u(t,\cdot), v(t,\cdot) \in H^1(D)$ and that $u, v \in \mathrm{Lip}([0,T];L^2(D))$: We note that in this case we can choose $\gamma = 1$ in (2.18) and (2.19) since the term containing v_0 vanishes.*

2.2. **Convergence rate for the one dimensional wave equation.** In the last section, we showed that the numerical scheme (2.6) converges to the weak solution of the 1-D wave equation. However, the key question is the rate at which the approximate solutions converge to the exact solution as the mesh is refined i.e, $\Delta x \to 0$. The answer to this question is provided in the following theorem,

Theorem 2.1. *Let $c \in C^{0,\alpha}(D)$ satisfy $\infty > \bar{c} \geq c(x) \geq \underline{c} > 0$ for all $x \in D$. Denote by (u,v) the solution of (2.2) and $(u_{\Delta x}, v_{\Delta x})$ the numerical approximation computed by the scheme (2.6) and defined in (2.20). Assume that the initial data $u_0, v_0 \in L^2(D)$ have moduli of continuity*

$$\nu_x^2(u_0,\sigma) \leq C\,\sigma^{2\gamma}, \quad \nu_x^2(v_0,\sigma) \leq C\,\sigma^{2\gamma}.$$

Then the approximation $(u_{\Delta x}(t,\cdot), v_{\Delta x}(t,\cdot))$ converges to the solution $(u(t,\cdot), v(t,\cdot))$, $0 < t < T$, and we have the estimate on the rate

$$\begin{aligned} &\|(u-u_{\Delta x})(t,\cdot)\|_{L^2(D)} + \|(v-v_{\Delta x})(t,\cdot)/c\|_{L^2(D)} \\ &\qquad \leq C\left(\|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/c\|_{L^2(D)} + \Delta x^{(\alpha\gamma)/(2(\alpha\gamma+1-\gamma))}\right), \end{aligned} \tag{2.24}$$

where C is a constant depending on c and T but not on Δx.

Proof. We let $\phi \in C_0^2((0,T) \times D)$ and define

$$\Lambda_T(u,v,k,\ell,\phi) := \int_{D_T} \left(\frac{(u-k)^2}{2} + \frac{(v-\ell)^2}{2c}\right)\phi_t - (u-k)(v-\ell)\phi_x \,dxdt \tag{2.25}$$

The above definition is an adaptation of the Kruzkhov doubling of variables technique [6] in our current L^2 setting.

For any even function $\omega(x) \in C_0^\infty(\mathbb{R})$ with the properties

$$0 \leq \omega \leq 1, \quad \omega(x) = 0 \text{ for } |x| \geq 1, \quad \int_{\mathbb{R}} \omega(x)\,dx,$$

we set

$$\omega_\epsilon(x) = \frac{1}{\epsilon}\omega\left(\frac{x}{\epsilon}\right),$$

and define for some $0 < \nu < \tau < T$,

$$\psi^\mu(t) := H_\mu(t-\nu) - H_\mu(t-\tau), \quad H_\mu(t) = \int_{-\infty}^t \omega_\mu(\xi)\,d\xi.$$

Then we define the function $\Omega : \Pi_T^2 \to \mathbb{R}$ by

$$\Omega(t,s,x,y) = \psi^\mu(t)\omega_{\epsilon_0}(t-s)\omega_\epsilon(x-y). \tag{2.26}$$

We assume without loss of generality $\Delta x \le \min\{\epsilon, \epsilon_0, \nu\}$. By the entropy inequality (2.22), we have for the solution (u,v) of (2.1) that $\Lambda_T(u,v,u_{\Delta x}(s,y),v_{\Delta x}(s,y),\phi) \ge 0$ for all $(s,y) \in D_T$ and test functions $\phi \in C_0^2((0,T)\times D)$. By (2.9), we have on the other hand that

(2.27)
$$\int_{D_T} \left(\frac{(u_{\Delta x} - u(t,x))^2}{2} + \frac{(v_{\Delta x} - v(t,x))^2}{2c} \right) D_s^- \phi - (u_{\Delta x} - u(t,x))(v_{\Delta x} - v(t,x)) D_y^c \phi \, dyds$$
$$\ge \int_{D_T} (v_{\Delta x} - v(t,x))^2 \left(\frac{1}{2c} - \frac{1}{2c_{\Delta x}} \right) D_s^- \phi \, dyds$$
$$- \frac{\Delta x^2}{2} (\theta - 1) \int_{D_T} \left(D_y^+ (u_{\Delta x} - u) D_y^+ (v_{\Delta x} - v) \right) D_y^+ \phi \, dyds$$
$$+ \frac{\Delta x}{4} \int_{D_T} (D_y^+ (v_{\Delta x} - v(t,x))^2 + D_y^+ (u_{\Delta x} - u(t,x))^2) D_y^+ \phi \, dyds$$

where $D_s^- \phi$ and $D_y^+ \phi$ have been defined in (**??**). Adding $\Lambda_T(u,v,u_{\Delta x}(s,y),v_{\Delta x}(s,y),\phi) \ge 0$ and (2.27), choosing Ω as a test function and integrating over D_T, we obtain

$$\underbrace{\int_{D_T^2} \left(\frac{(u_{\Delta x} - u)^2}{2} + \frac{(v_{\Delta x} - v)^2}{2c} \right) \left(\Omega_t + D_s^- \Omega \right) \, d\underline{z}}_{A} \tag{2.28}$$
$$- \underbrace{\int_{D_T^2} (u_{\Delta x} - u)(v_{\Delta x} - v) \left(\Omega_x + D_y^c \Omega \right) \, d\underline{z}}_{B}$$
$$\ge \underbrace{\int_{D_T^2} (v_{\Delta x} - v)^2 \left(\frac{1}{2c(x)} - \frac{1}{2c_{\Delta x}(y)} \right) D_s^- \Omega \, d\underline{z}}_{D}$$
$$+ \underbrace{\frac{\Delta x^2}{2} (\theta - 1) \int_{D_T^2} D_y^- \left[D_y^+ (u_{\Delta x} - u) \, D_y^+ (v_{\Delta x} - v) \right] \Omega \, d\underline{z}}_{E}$$
$$- \underbrace{\frac{\Delta x}{4} \int_{D_T^2} ((v_{\Delta x} - v(t,x))^2 + (u_{\Delta x} - u(t,x))^2) D_y^- D_y^+ \Omega \, d\underline{z}}_{F}$$

We rewrite the term A as

$$A = \int_{D_T^2} \eta(u - u_{\Delta x}, v - v_{\Delta x}, c)(\Omega_t + D_s^- \Omega) d\underline{z}$$
$$= \underbrace{\int_{D_T^2} \eta(u - u_{\Delta x}, v - v_{\Delta x}, c) \psi_t^\mu \omega_\epsilon \omega_{\epsilon_0} \, d\underline{z}}_{A_1}$$
$$+ \underbrace{\int_{D_T^2} \eta(u - u_{\Delta x}, v - v_{\Delta x}, c) \psi^\mu \, \omega_\epsilon \left(\partial_t \omega_{\epsilon_0} + D_s^- \omega_{\epsilon_0} \right) \, d\underline{z}}_{A_2}$$

The term A_1 can be written as

$$A_1 = \int_{D_T^2} \eta(u - u_{\Delta x}, v - v_{\Delta x}, c) \omega_\mu (t - \nu) \omega_\epsilon \omega_{\epsilon_0} \, d\underline{z} - \int_{D_T^2} \eta(u - u_{\Delta x}, v - v_{\Delta x}, c) \omega_\mu (t - \tau) \omega_\epsilon \omega_{\epsilon_0} \, d\underline{z}.$$

Introducing λ as

$$\lambda(t) = \int_0^T \int_{D^2} \eta\left(u_{\Delta x}(s,y) - u(t,x), v_{\Delta x}(s,y) - v(x,t), c(x)\right) \times \omega_\epsilon(x-y)\omega_{\epsilon_0}(t-s)\,dydxds, \tag{2.29}$$

we have that

$$A_1 = \int_0^T \lambda(t)\omega_\mu(t-\nu)\,dt - \int_0^T \lambda(t)\omega_\mu(t-\tau)\,dt,$$

so that (2.28) implies

$$\int_0^T \lambda(t)\omega_\mu(t-\nu)\,dt + |A_2| + |B| + |D| + |E| + |F| \geq \int_0^T \lambda(t)\omega_\mu(t-\tau)\,dt. \tag{2.30}$$

Our task is now to overestimate $|A_2|$, $|B|$, $|D|$, $|E|$ and $|F|$.

To estimate the term A_2, we note that

$$D_s^-\omega_{\epsilon_0} + \partial_t\omega_{\epsilon_0} = D_s^-\omega_{\epsilon_0} - \partial_s\omega_{\epsilon_0} = \frac{1}{\Delta t}\int_0^{\Delta t} (\xi - \Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,d\xi. \tag{2.31}$$

and observe that,

$$\frac{1}{\Delta t}\int_0^T \int_0^{\Delta t} \eta(u(t,x) - u_{\Delta x}(t,y), v(t,x) - v_{\Delta x}(t,y), c)(\xi - \Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,d\xi ds = 0,$$

since all the terms in the integrand except $\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)$ are independent of s. Therefore, subtracting this term from A_2, we obtain,

$$A_2 = \underbrace{\frac{1}{2\Delta t}\int_{D_T^2}\int_0^{\Delta t} (u_{\Delta x}(t,y) - u_{\Delta x})(2u - u_{\Delta x} - u_{\Delta x}(t,y))\psi^\mu\,\omega_\epsilon\,(\xi - \Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,d\xi d\underline{z}}_{A_{2,1}}$$
$$\underbrace{\frac{1}{2\Delta t}\int_{D_T^2}\int_0^{\Delta t} \frac{1}{c}(v_{\Delta x}(t,y) - v_{\Delta x})(2v - v_{\Delta x} - v_{\Delta x}(t,y))\psi^\mu\,\omega_\epsilon\,(\xi - \Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,d\xi d\underline{z}}_{A_{2,2}}$$

We will outline estimating the term $A_{2,1}$, the term $A_{2,2}$ is estimated in a similar way. By the triangle and Hölder's inequality

$$\begin{aligned}
|A_{2,1}| \leq & \frac{1}{2\Delta t}\int_{D_T^2}\int_0^{\Delta t} |u_{\Delta x}(t,y) - u_{\Delta x}(s,y)|\big(|u(t,x) - u_{\Delta x}(s,y)| + |u(t,x) - u_{\Delta x}(t,y)|\big) \\
& \times \psi^\mu\,\omega_\epsilon\,|\xi - \Delta t|\,|\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)|\,d\xi d\underline{z} \\
\leq & \frac{1}{2\Delta t}\int_0^{\Delta t}\int_0^T\int_0^T \left(\int_{D^2} |u_{\Delta x}(t,y) - u_{\Delta x}(s,y)|^2\omega_\epsilon\,dy\,dx\right)^{1/2} \\
& \times \left\{\left(\int_{D^2} |u(t,x) - u_{\Delta x}(s,y)|^2\omega_\epsilon\,dy\,dx\right)^{1/2}\right. \\
& \left. + \left(\int_{D^2} |u(t,x) - u_{\Delta x}(t,y)|^2\omega_\epsilon\,dy\,dx\right)^{1/2}\right\} \\
& \times \psi^\mu\,|\xi - \Delta t|\,|\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)|\,ds\,dtd\xi \\
\leq & \frac{1}{2\Delta t}\int_0^{\Delta t}\int_0^T \sup_{\substack{0\leq s\leq T \\ |t-s|<2\epsilon_0}} \left(\int_{D^2} |u_{\Delta x}(t,y) - u_{\Delta x}(s,y)|^2\omega_\epsilon\,dy\,dx\right)^{1/2} \\
& \times \left\{ \sup_{\substack{0\leq s\leq T \\ |t-s|<2\epsilon_0}} \left(\int_{D^2} |u(t,x) - u_{\Delta x}(s,y)|^2\omega_\epsilon\,dy\,dx\right)^{1/2}\right.
\end{aligned} \tag{2.32}$$

$$
\begin{aligned}
&\quad + \left(\int_{D^2} |u(t,x)-u_{\Delta x}(t,y)|^2 \omega_\epsilon \,dy\,dx\right)^{1/2}\Bigg\} \\
&\quad \times \psi^\mu\, |\xi-\Delta t| \int_0^T |\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)|\,ds\,dt\,d\xi \\
&\le \frac{C}{\Delta t\, \epsilon_0^{2-\gamma}} \int_0^{\Delta t}\int_0^T \Bigg\{ \sup_{\substack{0\le s\le T\\ |t-s|<2\epsilon_0}} \left(\int_{D^2} |u(t,x)-u_{\Delta x}(s,y)|^2\omega_\epsilon\,dy\,dx\right)^{1/2} \\
&\quad + \left(\int_{D^2} |u(t,x)-u_{\Delta x}(t,y)|^2\omega_\epsilon\,dy\,dx\right)^{1/2}\Bigg\}\psi^\mu\,|\xi-\Delta t|\,dt\,d\xi \\
&\le \frac{C\Delta t}{\epsilon_0^{2-\gamma}} \int_0^{\Delta t}\int_0^T \Bigg\{ \sup_{\substack{0\le s\le T\\ |t-s|<2\epsilon_0}} \left(\int_{D^2} |u(t,x)-u_{\Delta x}(s,y)|^2\omega_\epsilon\,dy\,dx\right)^{1/2} \\
&\quad + \left(\int_{D^2} |u(t,x)-u_{\Delta x}(t,y)|^2\omega_\epsilon\,dy\,dx\right)^{1/2}\Bigg\}\psi^\mu\ dt
\end{aligned}
$$

where we used the moduli of continuity for $u_{\Delta x}$, viz. (2.12), in the penultimate inequality and that $\Delta t \le \epsilon_0$.

$$
\begin{aligned}
(2.33)\qquad &\int_0^T \sup_{\substack{0\le s\le T\\ |t-s|<\epsilon_0}} \left(\int_{D^2} |u_{\Delta x}(s,y)-u(t,x)|^2\,\omega_\epsilon\,dydx\right)^{1/2}\psi^\mu\,dt \\
&\le \int_0^T \Bigg\{ \sup_{\substack{0\le s\le T\\ |t-s|<\epsilon_0}} \left(\int_{D^2} |u_{\Delta x}(s,y)-u_{\Delta x}(t,y)|^2\,\omega_\epsilon\,dydx\right)^{1/2} \\
&\quad + \left(\int_{D^2} |u_{\Delta x}(t,y)-u(t,x)|^2\,\omega_\epsilon\,dydx\right)^{1/2}\Bigg\}\psi^\mu\,dt \\
&\le CT\epsilon_0^\gamma + \int_0^T \left(\int_{D^2} |u_{\Delta x}(t,y)-u(t,x)|^2\,\omega_\epsilon\,dydx\right)^{1/2}\psi^\mu\,dt \\
&\le CT\epsilon_0^\gamma + \int_0^T \left(\int_0^T\int_{D^2} |u_{\Delta x}(s,y)-u(t,x)|^2\,\omega_\epsilon\omega_{\epsilon_0}\,dydxds\right)^{1/2} \\
&\quad + \int_0^T \left(\int_0^T\int_{D^2} |u_{\Delta x}(t,x)-u_{\Delta x}(s,x)|^2\,\omega_\epsilon\omega_{\epsilon_0}\,dydxds\right)^{1/2}\psi^\mu\,dt \\
&\le CT\epsilon_0^\gamma + \int_0^T \left(\int_0^T\int_{D^2} |u_{\Delta x}(s,y)-u(t,x)|^2\,\omega_\epsilon\omega_{\epsilon_0}\,dydxds\right)^{1/2}\psi^\mu\,dt
\end{aligned}
$$

by the triangle inequality and similarly

$$
\begin{aligned}
(2.34)\qquad &\int_0^T \sup_{\substack{0\le s\le T\\ |t-s|<\epsilon_0}} \left(\int_{D^2} \frac{1}{c}\,|v_{\Delta x}(s,y)-v(t,x)|^2\,\omega_\epsilon\,dydx\right)^{1/2}\psi^\mu\,dt \\
&\le \frac{CT\epsilon_0^\gamma}{\underline{c}} + \int_0^T \left(\int_0^T\int_{D^2} \frac{1}{c}\,|v_{\Delta x}(s,y)-v(t,x)|^2\,\omega_\epsilon\omega_{\epsilon_0}\,dydxds\right)^{1/2}\psi^\mu\,dt.
\end{aligned}
$$

Using λ, cf. (2.29), (2.32) can be bounded as

$$
|A_{2,2}| \le C\Delta t\,\epsilon_0^{2\gamma-2} + \frac{C\Delta t}{\epsilon_0^{2-\gamma}}\int_0^T \sqrt{\lambda(t)}\,\psi^\mu\,dt.
$$

and so, using a similar argument for the term $A_{2,1}$

$$|A_2| \le C\Delta t\, \epsilon_0^{2\gamma-2} + \frac{C\Delta t}{\epsilon_0^{2-\gamma}} \int_0^T \sqrt{\lambda(t)}\, \psi^\mu \, dt. \tag{2.35}$$

In order to bound the term B, we use

$$\begin{aligned}\Omega_x + D_y^c\Omega &= \frac{-1}{4\Delta x}\int_0^{\Delta x} (\xi-\Delta x)^2 \left[\partial_{yyy}\Omega(t,s,x,y-\xi) + \partial_{yyy}\Omega(t,s,x,y+\xi)\right] d\xi \\ &= \frac{1}{4\Delta x}\int_0^{\Delta x} (\xi-\Delta x)^2 \left[\partial_{xxx}\Omega(t,s,x,y-\xi) + \partial_{xxx}\Omega(t,s,x,y+\xi)\right] d\xi.\end{aligned}$$

and that

$$\begin{aligned}\frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2} &(\xi-\Delta x)^2 \left(u_{\Delta x} - u(t,y)\right)\left(v_{\Delta x} - v(t,y)\right) \\ &\times \left[\partial_{xxx}\omega_\epsilon(x-y+\xi) + \partial_{xxx}\omega_\epsilon(x-y-\xi)\right]\omega_{\epsilon_0}\psi^\mu \, d\xi d\underline{z} = 0,\end{aligned}$$

since all the terms in the integrand, except $[\partial_{xxx}\omega_\epsilon(x-y+\xi) + \partial_{xxx}\omega_\epsilon(x-y-\xi)]$, are independent of x. We subtract this term from B and add and subtract the term

$$\begin{aligned}\frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2} &(\xi-\Delta x)^2 \left(u_{\Delta x} - u(t,y)\right)\left(v_{\Delta x} - v(t,x)\right) \\ &\times \left[\partial_{xxx}\omega_\epsilon(x-y+\xi) + \partial_{xxx}\omega_\epsilon(x-y-\xi)\right]\omega_{\epsilon_0}\psi^\mu \, d\xi d\underline{z}\end{aligned}$$

so that

$$\begin{aligned}B &= \frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2} (\xi-\Delta x)^2 \left(u(t,y) - u(t,x)\right)\left(v_{\Delta x} - v(t,x)\right) \\ &\qquad \times \left[\partial_{xxx}\omega_\epsilon(x-y+\xi) + \partial_{xxx}\omega_\epsilon(x-y-\xi)\right]\omega_{\epsilon_0}\psi^\mu \, d\xi d\underline{z} \\ &\quad + \frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2} (\xi-\Delta x)^2 \left(u_{\Delta x} - u(t,y)\right)\left(v(t,y) - v(t,x)\right) \\ &\qquad \times \left[\partial_{xxx}\omega_\epsilon(x-y+\xi) + \partial_{xxx}\omega_\epsilon(x-y-\xi)\right]\omega_{\epsilon_0}\psi^\mu \, d\xi d\underline{z} \\ &:= B_1 + B_2.\end{aligned}$$

We start by bounding B_1,

$$\begin{aligned}|B_1| &\le \frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2} (\xi-\Delta x)^2 |u(t,y) - u(t,x)|\, |v_{\Delta x} - v(t,x)| \\ &\qquad \times \left|\partial_{xxx}\omega_\epsilon(x-y+\xi) + \partial_{xxx}\omega_\epsilon(x-y-\xi)\right|\omega_{\epsilon_0}\psi^\mu \, d\xi d\underline{z} \\ &\le \frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T}\left(\int_{D_T} |u(t,y)-u(t,x)|^2 \omega_{\epsilon_0}\, dy\, ds\right)^{1/2} \\ &\qquad \times \left(\int_{D_T} |v_{\Delta x} - v(t,x)|^2\omega_{\epsilon_0}\, dy\, ds\right)^{1/2} (\xi-\Delta x)^2 \\ &\qquad \times \left|\partial_{xxx}\omega_\epsilon(x-y+\xi) + \partial_{xxx}\omega_\epsilon(x-y-\xi)\right| \psi^\mu \, dx\, dt\, d\xi \\ &\le \frac{1}{4\Delta x}\int_0^{\Delta x}\int_0^T \sup_{\substack{x \text{ s.t.} \\ |x-y|\le 3\epsilon}} \left(\int_{D_T} |u(t,y)-u(t,x)|^2 \omega_{\epsilon_0}\, dy\, ds\right)^{1/2} \\ &\qquad \times \sup_{\substack{x \text{ s.t.} \\ |x-y|\le 3\epsilon}} \left(\int_{D_T} |v_{\Delta x} - v(t,x)|^2\omega_{\epsilon_0}\, dy\, ds\right)^{1/2} (\xi-\Delta x)^2 \\ &\qquad \times \int_D \left|\partial_{xxx}\omega_\epsilon(x-y+\xi) + \partial_{xxx}\omega_\epsilon(x-y-\xi)\right| dx\, \psi^\mu \, dt\, d\xi\end{aligned}$$

$$\le \frac{C}{\epsilon^{3-\gamma}\Delta x}\int_0^{\Delta x}\int_0^T \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon}} \left(\int_{D_T} |v_{\Delta x} - v(t,x)|^2 \omega_{\epsilon_0}\, dy\, ds\right)^{1/2} (\xi - \Delta x)^2 \psi^\mu \, dt\, d\xi$$
$$\le \frac{C\Delta x^2}{\epsilon^{3-\gamma}}\int_0^T \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon}} \left(\int_{D_T} |v_{\Delta x} - v(t,x)|^2 \omega_{\epsilon_0}\, dy\, ds\right)^{1/2} \psi^\mu \, dt$$

where we have used that ω_ϵ is compactly supported in $[-\epsilon,\epsilon]$, and where C is a constant depending on the L^2-norms and the moduli of continuity of the initial data and on T. Using that (c.f. (2.33))

$$\int_0^T \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon}} \left(\int_{D_T} |u_{\Delta x}(s,y) - u(t,x)|^2\, \omega_{\epsilon_0}\, dyds\right)^{1/2} \psi^\mu \, dt \tag{2.36}$$
$$\le \int_0^T \Bigg\{ \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon}} \left(\int_{D_T} |u(t,y) - u(t,x)|^2\, \omega_{\epsilon_0}\, dyds\right)^{1/2}$$
$$+ \left(\int_{D_T} |u_{\Delta x}(s,y) - u(t,y)|^2\, \omega_{\epsilon_0}\, dyds\right)^{1/2} \Bigg\} \psi^\mu \, dt$$
$$\le CT\epsilon^\gamma + \int_0^T \left(\int_{D_T} |u_{\Delta x}(s,y) - u(t,y)|^2\, \omega_{\epsilon_0}\, dyds\right)^{1/2} \psi^\mu \, dt$$
$$\le CT\epsilon^\gamma + \int_0^T \left(\int_0^T \int_{D^2} |u_{\Delta x}(s,y) - u(t,x)|^2\, \omega_\epsilon \omega_{\epsilon_0}\, dydxds\right)^{1/2} \psi^\mu \, dt$$
$$+ \int_0^T \left(\int_0^T \int_{D^2} |u(t,y) - u(t,x)|^2\, \omega_\epsilon\, dydx\right)^{1/2} \psi^\mu \, dt$$
$$\le CT\epsilon^\gamma + \int_0^T \left(\int_0^T \int_{D^2} |u_{\Delta x}(s,y) - u(t,x)|^2\, \omega_\epsilon \omega_{\epsilon_0}\, dydxds\right)^{1/2} \psi^\mu \, dt,$$

and analogously,

$$\int_0^T \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon}} \left(\int_{D^T} |v_{\Delta x}(s,y) - v(t,x)|^2\, \omega_{\epsilon_0}\, dyds\right)^{1/2} \psi^\mu \, dt \tag{2.37}$$
$$\le CT\epsilon^\gamma + \int_0^T \left(\int_0^T \int_{D^2} |v_{\Delta x}(s,y) - v(t,x)|^2\, \omega_\epsilon \omega_{\epsilon_0}\, dydxds\right)^{1/2} \psi^\mu \, dt,$$

for B_1 we obtain the estimate

$$|B_1| \le \frac{C\Delta x^2}{\epsilon^{3-2\gamma}} + \frac{C\Delta x^2}{\epsilon^{3-\gamma}}\int_0^T \sqrt{\lambda(t)}\,\psi^\mu\, dt. \tag{2.38}$$

Similarly

$$|B_2| \le \frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2} (\xi-\Delta x)^2 |u_{\Delta x} - u(t,y)|\, |v(t,y) - v(t,x)|$$
$$\times |\partial_{xxx}\omega_\epsilon(x-y+\xi) + \partial_{xxx}\omega_\epsilon(x-y-\xi)|\, \omega_{\epsilon_0}\psi^\mu\, d\xi d\underline{z}$$
$$\le \frac{C\Delta x^2}{\epsilon^{3-\gamma}}\int_0^T \left(\int_{D_T} |v_{\Delta x}(s,y) - v(t,y)|^2 \omega_{\epsilon_0}\, dy\, ds\right)^{1/2} \psi^\mu\, dt$$

Using (2.37), we find, as for B_1,

$$|B_2| \le \frac{C\Delta x^2}{\epsilon^{3-2\gamma}} + \frac{C\Delta x^2}{\epsilon^{3-\gamma}}\int_0^T \sqrt{\lambda(t)}\,\psi^\mu\, dt, \tag{2.39}$$

and therefore

$$|B| \le \frac{C\Delta x^2}{\epsilon^{3-2\gamma}} + \frac{C\Delta x^2}{\epsilon^{3-\gamma}}\int_0^T \sqrt{\lambda(t)}\,\psi^\mu\, dt. \tag{2.40}$$

We proceed to bounding the term D. Observing that

$$\int_{D_T^2} (v(t,x) - v_{\Delta x}(t,y))^2 \left(\frac{1}{2c(x)} - \frac{1}{2c_{\Delta x}(y)} \right) D_s^- \Omega \, d\underline{z} = 0,$$

we can rewrite D as

$$D = \int_{D_T^2} \left((v(t,x) - v_{\Delta x}(t,y))^2 - (v(t,x) - v_{\Delta x}(s,y))^2 \right) \left(\frac{1}{2c(x)} - \frac{1}{2c_{\Delta x}(y)} \right) D_s^- \Omega \, d\underline{z}.$$

Noting that we note that,

$$D_s^- \Omega(t,s,x,y) = \frac{1}{\Delta t} \int_0^{\Delta t} \Omega_s(t, s-\xi, x, y) \, d\xi, \tag{2.41}$$

this becomes

$$\begin{aligned} D = \frac{1}{\Delta t} \int_{D_T^2} \int_0^{\Delta t} & (2v(t,x) - v_{\Delta x}(t,y) - v_{\Delta x}(s,y)) \\ & \times (v_{\Delta x}(t,y) - v_{\Delta x}(s,y)) \frac{c_{\Delta x}(y) - c(x)}{2c(x) c_{\Delta x}(y)} \Omega_s \, d\xi \, d\underline{z}. \end{aligned}$$

which can be bounded by

$$\begin{aligned} |D| &\leq \frac{1}{2\underline{c}\Delta t} \sup_{|x-y|<\epsilon} |c(x) - c_{\Delta x}(y)| \\ & \quad \times \int_{D_T^2} \int_0^{\Delta t} \frac{1}{c} \left| 2v(t,x) - v_{\Delta x}(t,y) - v_{\Delta x}(s,y) \right| \left| v_{\Delta x}(t,y) - v_{\Delta x}(s,y) \right| |\Omega_s| \, d\xi \, d\underline{z} \\ & \leq \frac{C(\epsilon + \Delta x)^\alpha}{2\underline{c}\,\epsilon_0} \sup_{t \in (0,T)} \nu_t^2 (v_{\Delta x}(t, \cdot), \epsilon_0)^{1/2} \\ & \quad \times \int_0^T \sup_{\substack{0 \leq s \leq T \\ |t-s|<\epsilon_0}} \left(\int_{D^2} \frac{1}{c} \left| v_{\Delta x}(t,y) - v(s,x) \right|^2 \omega_\epsilon \, dy dx \right)^{1/2} \psi^\mu \, dt \\ & \leq \frac{C(\epsilon + \Delta x)^\alpha}{2\underline{c}\epsilon_0^{1-2\gamma}} + \frac{C(\epsilon + \Delta x)^\alpha}{2\underline{c}\epsilon_0^{1-\gamma}} \int_0^T \sqrt{\lambda(t)} \, \psi^\mu \, dt \end{aligned} \tag{2.42}$$

where we have used (2.34) for the last inequality. For the term E, we note that it can be written

$$E = \frac{\Delta x^2}{2} (\theta - 1) \int_0^T \int_{D_T} D_y^- \left[D_y^+ u_{\Delta x} D_y^+ v_{\Delta x} \right] \int_D \omega_\epsilon (x-y) \, dx \, \omega_{\epsilon_0} \psi^\mu \, dy \, ds \, dt,$$

so that

$$\begin{aligned} E &= \frac{\Delta x^2}{2} (\theta - 1) \int_0^T \int_{D_T} D_y^- \left[D_y^+ u_{\Delta x} D_y^+ v_{\Delta x} \right] \omega_{\epsilon_0} \psi^\mu \, dy \, ds \, dt, \\ &= \frac{\Delta x^3}{2} (\theta - 1) \int_0^T \int_0^T \sum_j D_y^- \left[D_y^+ u_{\Delta x}(s, x_j) D_y^+ v_{\Delta x}(s, x_j) \right] \omega_{\epsilon_0} \psi^\mu \, ds \, dt. \\ &= 0 \end{aligned} \tag{2.43}$$

In order to estimate the term F, we use that

$$D_x^+ D_x^- \phi(x) = \frac{1}{2\Delta x^2} \int_{-\Delta x}^0 \int_0^{\Delta x} \phi''(x + \eta + \xi) \, d\xi \, d\eta, \tag{2.44}$$

and that

$$\frac{1}{8\Delta x} \int_{-\Delta x}^0 \int_0^{\Delta x} \int_{D_T^2} ((v_{\Delta x} - v(t,y))^2 + (u_{\Delta x} - u(t,y))^2) \partial_x^2 \omega_\epsilon (x - y - \eta - \xi) \, \omega_{\epsilon_0} \psi^\mu d\underline{z} \, d\xi \, d\eta = 0,$$

since all the terms in the integrand, but $\partial_x^2 \omega_\epsilon(x-y-\eta-\xi)$ are independent of x. We subtract this term from F to find

$$F = \underbrace{\frac{1}{8\Delta x}\int_{-\Delta x}^{0}\int_{0}^{\Delta x}\int_{D_T^2}(v-v(t,y))(v+v(t,y)-2v_{\Delta x})\partial_x^2\omega_\epsilon(x-y-\eta-\xi)\,\omega_{\epsilon_0}\psi^\mu d\underline{z}\,d\xi\,d\eta}_{F_1}$$
$$+\underbrace{\frac{1}{8\Delta x}\int_{-\Delta x}^{0}\int_{0}^{\Delta x}\int_{D_T^2}(u-u(t,y))(u+u(t,y)-2u_{\Delta x})\partial_x^2\omega_\epsilon(x-y-\eta-\xi)\,\omega_{\epsilon_0}\psi^\mu d\underline{z}\,d\xi\,d\eta}_{F_2}.$$

The integrals F_1 and F_2 are estimated in the same way, therefore we outline only the estimate of F_1.

$$\begin{aligned}
|F_1| &\le \frac{1}{8\Delta x}\int_{-\Delta x}^{0}\int_{0}^{\Delta x}\int_{D_T^2}|v-v(t,y)|\big(|v-v_{\Delta x}|+|v(t,y)-v_{\Delta x}|\big)|\partial_x^2\omega_\epsilon|\,\omega_{\epsilon_0}\psi^\mu d\underline{z}\,d\xi\,d\eta\\
&\le \frac{1}{8\Delta x}\int_{-\Delta x}^{0}\int_{0}^{\Delta x}\int_0^T \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon}}\left(\int_{D_T}|v-v(t,y)|^2\omega_{\epsilon_0}\,dy\,ds\right)^{1/2}\\
&\qquad\times\left\{\sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon}}\left(\int_{D_T}|v-v_{\Delta x}|^2\omega_{\epsilon_0}\,dy\,ds\right)^{1/2}\right.\\
&\qquad\qquad\left.+\left(\int_{D_T}|v(t,y)-v_{\Delta x}|^2\omega_{\epsilon_0}\,dy\,ds\right)^{1/2}\right\}\\
&\qquad\int_D|\partial_x^2\omega_\epsilon|\,dx\,\psi^\mu\,dt d\xi\,d\eta\\
&\le \frac{C\Delta x}{\epsilon^{2-\gamma}}\int_0^T\left\{\sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon}}\left(\int_{D_T}|v-v_{\Delta x}|^2\omega_{\epsilon_0}\,dy\,ds\right)^{1/2}\right.\\
&\qquad\left.+\left(\int_{D_T}|v(t,y)-v_{\Delta x}|^2\omega_{\epsilon_0}\,dy\,ds\right)^{1/2}\right\}\psi^\mu\,dt
\end{aligned}$$

Using (2.37), we find

$$|F_1| \le \frac{C\Delta x}{\epsilon^{2-2\gamma}} + \frac{C\Delta x}{\epsilon^{2-\gamma}}\int_0^T\sqrt{\lambda(t)}\psi^\mu\,dt$$

and therefore

$$|F| \le \frac{C\Delta x}{\epsilon^{2-2\gamma}} + \frac{C\Delta x}{\epsilon^{2-\gamma}}\int_0^T\sqrt{\lambda(t)}\,\psi^\mu\,dt. \tag{2.45}$$

Referring to (2.30), we have established the following bounds

$$\begin{aligned}
|A_2| &\le C\left(\frac{\Delta x}{\epsilon_0^{2-2\gamma}}+\frac{\Delta x}{\epsilon_0^{2-\gamma}}\int_0^T\sqrt{\lambda(t)}\,\psi^\mu\,dt\right),\\
|B| &\le C\left(\frac{\Delta x^2}{\epsilon^{3-2\gamma}}+\frac{\Delta x^2}{\epsilon^{3-\gamma}}\int_0^T\sqrt{\lambda(t)}\,\psi^\mu\,dt\right),\\
|D| &\le C\left(\frac{\epsilon^\alpha}{\epsilon_0^{1-2\gamma}}+\frac{\epsilon^\alpha}{\epsilon_0^{1-\gamma}}\int_0^T\sqrt{\lambda(t)}\,\psi^\mu\,dt\right),\\
|E| &= 0,\\
|F| &\le C\left(\frac{\Delta x}{\epsilon^{2-2\gamma}}+\frac{\Delta x}{\epsilon^{2-\gamma}}\int_0^T\sqrt{\lambda(t)}\,\psi^\mu\,dt\right),
\end{aligned}$$

where we have used that $\Delta t = C\Delta x$ and $\Delta x \le \epsilon$. Hence,

$$\int_0^T \lambda(t)\omega_\mu(t-\tau)\,dt \le \int_0^T \lambda(t)\omega_\mu(t-\nu)\,dt + \underbrace{C\left(\frac{\Delta x}{\epsilon_0^{2-2\gamma}} + \frac{\Delta x^2}{\epsilon^{3-2\gamma}} + \frac{\epsilon^\alpha}{\epsilon_0^{1-2\gamma}} + \frac{\Delta x}{\epsilon^{2-2\gamma}}\right)}_{M_1}$$

$$+ \underbrace{C\left(\frac{\Delta x}{\epsilon_0^{2-\gamma}} + \frac{\Delta x^2}{\epsilon^{3-\gamma}} + \frac{\epsilon^\alpha}{\epsilon_0^{1-\gamma}} + \frac{\Delta x}{\epsilon^{2-\gamma}}\right)}_{M_2} \int_0^T \sqrt{\lambda(t)}\,\psi^\mu\,dt.$$

Sending μ to zero, we find

$$\lambda(\tau) \le \lambda(\nu) + M_1 + M_2 \int_\nu^\tau \sqrt{\lambda(t)}\,dt.$$

With an application of a Gronwall type inequality, [3, Chapter 1, Theorem 4], we obtain the estimate

$$\lambda(\tau) \le \left(\sqrt{\lambda(\nu) + M_1} + (\tau-\nu)M_2\right)^2 \le 2\left(\lambda(\nu) + M_1 + T^2 M_2^2\right). \tag{2.46}$$

By the triangle inequality, we have

(2.47)

$$\begin{aligned}
&\left|\left(\int_D\int_{D_T} |u_{\Delta x}(s,x) - u(t,y)|^2\,\omega_\epsilon\omega_{\epsilon_0}\,dxdsdy\right)^{1/2} - \|u(t,\cdot) - u_{\Delta x}(t,\cdot)\|_{L^2(D)}\right| \\
&\le \left(\int_D\int_{D_T} |u_{\Delta x}(s,x) - u_{\Delta x}(t,y)|^2\,\omega_\epsilon\omega_{\epsilon_0}\,dxdsdy\right)^{1/2} \\
&\le \left(\int_{D^2} |u_{\Delta x}(t,x) - u_{\Delta x}(t,y)|^2\,\omega_\epsilon\,dxdy\right)^{1/2} + \left(\int_{D_T} |u_{\Delta x}(t,x) - u_{\Delta x}(s,x)|^2\,\omega_{\epsilon_0}\,dsdx\right)^{1/2} \\
&\le C\,(\epsilon_0^\gamma + \epsilon^\gamma),
\end{aligned}$$

and similarly

$$\left|\left(\int_D\int_{D_T} \frac{1}{c(x)}|v_{\Delta x}(t,x) - v(s,y)|^2\,\omega_\epsilon\omega_{\epsilon_0}\,dxdsdy\right)^{1/2} - \|(v - v_{\Delta x})(t,\cdot)/c\|_{L^2(D)}\right| \le C\,(\epsilon_0^\gamma + \epsilon^\gamma). \tag{2.48}$$

Moreover,

$$\begin{aligned}
&\|(u - u_{\Delta x})(\nu,\cdot)\|_{L^2(D)} + \|(v - v_{\Delta x})(\nu,\cdot)/c\|_{L^2(D)} \\
&\quad\le \|u_{\Delta x}(\nu,\cdot) - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_{\Delta x}(\nu,\cdot) - v_{\Delta x}(0,\cdot))/c\|_{L^2(D)} \\
&\qquad + \|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/c\|_{L^2(D)} \\
&\qquad + \|u(\nu,\cdot) - u_0\|_{L^2(D)} + \|(v(\nu,\cdot) - v_0)/c\|_{L^2(D)} \\
&\quad\le C(\nu + \Delta t)^\gamma + \|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/c\|_{L^2(D)}.
\end{aligned} \tag{2.49}$$

Write

$$e(\tau) = \|(u - u_{\Delta x})(\tau,\cdot)\|_{L^2(D)} + \|(v - v_{\Delta x})(\tau,\cdot)/c\|_{L^2(D)}.$$

Thus, combining (2.46), (2.47), (2.48) and (2.49), the definition of M_1 and M_2 and some basic calculus inequalities, we obtain

$$\begin{aligned}
e^2(\tau) \le C\Big(&e^2(0) + \epsilon^{2\gamma} + \epsilon_0^{2\gamma} + \frac{\Delta x}{\epsilon_0^{2-2\gamma}} + \frac{\epsilon^\alpha}{\epsilon_0^{1-2\gamma}} + \frac{\Delta x^2}{\epsilon_0^{4-2\gamma}} \\
&+ \frac{\Delta x^4}{\epsilon^{6-2\gamma}} + \frac{\epsilon^{2\alpha}}{\epsilon_0^{2(1-\gamma)}} + \frac{\Delta x^2}{\epsilon^{3-2\gamma}} + \frac{\Delta x}{\epsilon^{2-2\gamma}} + \frac{\Delta x^2}{\epsilon^{4-2\gamma}}\Big).
\end{aligned} \tag{2.50}$$

Hence, choosing $\epsilon = \epsilon_0^{1/\alpha}$ and $\epsilon = \Delta x^{1/(2(\gamma\alpha+1-\gamma))}$,

$$e(\tau) \leq C\left(e(0) + \Delta x^{(\alpha\gamma)/(2(\alpha\gamma+1-\gamma))}\right).$$

□

This enables us to get a rate for the approximation $r_{\Delta x} := v_{\Delta x}/c_{\Delta x}$ to q, the second variable of the solution of (2.1):

Corollary 2.1. *Under the assumptions of Lemma 2.1, the approximation $(u_{\Delta x}, r_{\Delta x})$ converges to the solution of (2.1) at the rate*

$$\begin{aligned} &\|u(t,\cdot) - u_{\Delta x}(t,\cdot)\|_{L^2(D)} + \|r(t,\cdot) - r_{\Delta x}(t,\cdot)\|_{L^2(D)} \\ &\quad \leq C_1\left(\Delta x^{(\alpha\gamma)/(2(\alpha\gamma+1-\gamma))} + \|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/c\|_{L^2(D)}\right) \\ &\quad\quad + C_2\Delta x^{\alpha}, \end{aligned} \tag{2.51}$$

for $0 < t < T$ where C_1 is a constant depending on $\underline{c}$, $\overline{c}$, $\|c\|_{C^{0,\alpha}}$ and T but not on Δx, and C_2 a constant depending on the L^2-norm of v_0 and on $\underline{c}$, $\overline{c}$ and $\|c\|_{C^{0,\alpha}}$.

Proof. The result follows upon noting that

$$\begin{aligned} \|r(t,\cdot) - r_{\Delta x}(t,\cdot)\|_{L^2(D)} &\leq \frac{1}{\underline{c}}\left(\int |v(t,x) - v_{\Delta x}(t,x)|^2\, dx\right)^{1/2} \\ &\quad + \frac{1}{\underline{c}}\left(\int |c(x) - c_{\Delta x}(x)|^2 |r(t,x)|^2\, dx\right)^{1/2} \\ &\leq \frac{1}{\underline{c}}\|v(t,\cdot) - v_{\Delta x}(t,\cdot)\|_{L^2(D)} \\ &\quad + \frac{\|c\|_{C^{0,\alpha}}}{\underline{c}}\left(\|r_0\|_{L^2(D)} + \|u_0\|_{L^2(D)}\right)\Delta x^{\alpha}, \end{aligned}$$

and using the result from Lemma 2.1. □

2.2.1. *Approximation of the solution p of the wave equation (1.1) in one space dimension.* Finally, we would like to approximate the solution p of the original second order wave equation (1.1) by using the approximated solutions of the first order wave equation (2.1). We thus define discrete quantities via,

$$D_t^+ p_j^n = u_j^n, \quad j \in \mathbb{Z}, n = 1, \dots, N, \tag{2.52a}$$

$$p_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} p_0(x)\, dx, \quad j \in \mathbb{Z}, \tag{2.52b}$$

and then define $p_{\Delta x}$ via a piecewise constant or piecewise linear interpolation of p_j^n on the grid. We can rewrite (2.6) as a scheme for p_j^n:

$$D_t^+ D_t^+ p_j^n = D_x^c v_j^n + \frac{\Delta x}{2} D_x^+ D_x^- D_t^+ p_j^n, \tag{2.53a}$$

$$\frac{D_t^+ v_j^n}{c_j} = D_x^c D_t^+ p_j^n + \frac{\Delta x}{2} D_x^+ D_x^- v_j^n, \quad j \in \mathbb{Z},\ n = 1, \dots, N, \tag{2.53b}$$

From Lemma 2.1, we find that

$$\begin{aligned} &\Delta x \sum_j |D_t^+ p_j^n|^2, \Delta x \sum_j |D_{\gamma,t}^+ D_t^+ p_j^n|^2, \Delta x \sum_j |D_{\gamma,x}^+ D_t^+ p_j^n|^2, \\ &\qquad\qquad \Delta x \sum_j |D_{\gamma,t}^+ D_x^+ p_j^n|^2 \leq C(u_0, v_0, w_0), \quad n = 1, \dots, N. \end{aligned}$$

We would like to find a bound on $\Delta x \sum_j |D_x^+ p_j^n|^2$ as well and $\Delta x \sum_j |D_{\gamma,x}^+ D_x^+ p_j^n|^2$ and then show that in the limit $\lim_{\Delta x \to 0} D_x^+ p_{\Delta x} = p_x = v/c$ so that the limit $\lim_{\Delta x \to 0} p_{\Delta x} = p$ is the unique solution of (1.1). We therefore sum the scheme (2.53) over $m = 0, \dots, n-1$ and multiply by Δt to obtain,

$$
\begin{aligned}
D_t^x p_j^n - D_t^+ p_j^0 &= \Delta t \sum_{m=0}^{n-1} D_x^c v_j^m + \frac{\Delta x}{2} D_x^+ D_x^- p_j^n - \frac{\Delta x}{2} D_x^+ D_x^- p_j^0, \\
\frac{v_j^n}{c_j} - \frac{v_j^0}{c_j} &= D_x^c p_j^n - D_x^c p_j^0 + \frac{\Delta x \Delta t}{2} \sum_{m=0}^{n-1} D_x^+ D_x^- v_j^m,
\end{aligned} \tag{2.54}
$$

Applying the operator $D \in \{\mathrm{Id}, D_{\beta,x}^{\pm}, 0 \le \beta \le \min\{\alpha,\gamma\}\}$ to both equations in (2.54), squaring them, adding and summing over j, we obtain

$$
\begin{aligned}
&\sum_j \left(|D D_t^+ p_j^n|^2 + |D(v_j^n/c_j)|^2\right) \\
&= \Delta t^2 \sum_j \Big|\sum_{m=0}^{n-1} D D_x^c v_j^m\Big|^2 + \frac{\Delta x^2}{4} \sum_j |D D_x^+ D_x^- p_j^n|^2 + \sum_j |D D_x^c p_j^n|^2 \\
&\quad + \frac{\Delta x^2 \Delta t}{4} \sum_j \Big|\sum_{m=0}^{n-1} D D_x^+ D_x^- v_j^m\Big|^2 + \sum_j \left(D\left(D_t^+ p_j^0 - \frac{\Delta x}{2} D_x^+ D_x^- p_j^0\right)\right)^2 \\
&\quad + \sum_j \left(D\left(v_j^0/c_j - D_x^c p_j^0\right)\right)^2 + \underbrace{\Delta x \Delta t \sum_j \sum_{m=0}^{n-1} (D D_x^c v_j^m)(D D_x^+ D^- p_j^n)}_{\text{ä}} \\
&\quad \underbrace{+ 2\sum_j \left(D\left(D_t^+ p_j^0 - \frac{\Delta x}{2} D_x^+ D_x^- p_j^0\right)\right)\left(\Delta t \sum_{m=0}^{n-1} D D_x^c v_j^m + \frac{\Delta x}{2} D D_x^+ D_x^- p_j^n\right)}_{\text{å}} \\
&\quad \underbrace{+ \Delta x \Delta t \sum_j \sum_{m=0}^{n-1} (D D_x^c p_j^n)(D D_x^+ D_x^- v_j^m)}_{\text{ü}} \\
&\quad \underbrace{+ 2\sum_j \left(D\left(v_j^0/c_j - D_x^c p_j^0\right)\right)\left(D D_x^c p_j^n + \frac{\Delta x \Delta t}{2} \sum_{m=0}^{n-1} D D_x^+ D_x^- v_j^m\right)}_{\text{ø}}.
\end{aligned} \tag{2.55}
$$

We note that

$$
\text{ä} + \text{ü} = 0,
$$

obtained by summing by parts three times. Moreover, we have for the terms **ø** and **å**, for any $\delta > 0$,

$$
\begin{aligned}
|\text{å}| &\le \frac{2}{\delta} \sum_j \left(D\left(D_t^+ p_j^0 - \frac{\Delta x}{2} D_x^+ D_x^- p_j^0\right)\right)^2 + \delta \Delta t^2 \sum_j \Big|\sum_{m=0}^{n-1} D D_x^c v_j^m\Big|^2 + \frac{\Delta x^2 \delta}{4} \sum_j |D D_x^+ D_x^- p_j^n|^2 \\
|\text{ø}| &\le \frac{2}{\delta} \sum_j \left(D\left(v_j^0/c_j - D_x^c p_j^0\right)\right)^2 + \frac{\delta \Delta t^2 \Delta x^2}{4} \sum_j \Big|\sum_{m=0}^{n-1} D D_x^+ D_x^- v_j^m\Big|^2 + \delta \sum_j |D D_x^c p_j^n|^2.
\end{aligned}
$$

Choosing $\delta = 1/2$, and rearranging terms in (2.55), we find

$$
\begin{aligned}
&2\Delta x \sum_j \left(|D D_t^+ p_j^n|^2 + |D(v_j^n/c_j)|^2\right) + 6\Delta x \sum_j \left(D\left(v_j^0/c_j - D_x^c p_j^0\right)\right)^2 \\
&\quad + 6\Delta x \sum_j \left(D\left(D_t^+ p_j^0 - \frac{\Delta x}{2} D_x^+ D_x^- p_j^0\right)\right)^2
\end{aligned} \tag{2.56}
$$

$$\geq \Delta x \Delta t^2 \sum_j \Big| \sum_{m=0}^{n-1} DD_x^c v_j^m \Big|^2 + \frac{\Delta x^3}{4} \sum_j |DD_x^+ D_x^- p_j^n|^2 + \Delta x \sum_j |DD_x^c p_j^n|^2$$
$$+ \frac{\Delta x^3 \Delta t^2}{4} \sum_j \Big| \sum_{m=0}^{n-1} DD_x^+ D_x^- v_j^m \Big|^2.$$

By Lemma 2.1, we know that the right hand side of the above equation is bounded and therefore the right hand side as well. Moreover, since

$$\frac{\Delta x^3}{4} \sum_j |DD_x^+ D_x^- p_j^n|^2 + \Delta x \sum_j |DD_x^c p_j^n|^2 = \frac{\Delta x}{2} \sum_j \left(|DD_x^- p_j^n|^2 + |DD_x^+ p_j^n|^2 \right),$$

we obtain the desired bounds on $\Delta x \sum_j |D_x^+ p_j^n|^2$ and $\Delta x \sum_j |D_{\beta,x}^+ D_x^+ p_j^n|^2$. Hence $p_{\Delta x}$ converges to a limit function p in $H^{1+s}([0,T] \times D) \cap C([0,T]; H^{1+s}(D))$ for all $0 \leq s < \beta$. Using that also

$$\frac{\Delta x^3 \Delta t^2}{4} \sum_j \Big| \sum_{m=0}^{n-1} DD_x^+ D_x^- v_j^m \Big|^2 + \Delta x \Delta t^2 \sum_j \Big| \sum_{m=0}^{n-1} DD_x^c v_j^m \Big|^2 = \Delta x \Delta t^2 \sum_j \Big| \sum_{m=0}^{n-1} DD_x^- v_j^m \Big|^2, \tag{2.57}$$

we obtain by the second equation in (2.54), that

$$\begin{aligned}
\Delta x \sum_j \Big| \frac{v_j^n}{c_j} - D_x^c p_j^n \Big|^2 &\leq 2\Delta x \sum_j \Big| \frac{v_j^0}{c_j} - D_x^c p_j^0 \Big|^2 + \frac{\Delta x^3 \Delta t^2}{2} \sum_j \Big| \sum_{m=0}^{n-1} D_x^+ D_x^- v_j^m \Big|^2 \\
&\leq C \Delta x^{2\beta} + \frac{\Delta x^{2\beta+1} \Delta t^2}{2} \sum_j \Big| \sum_{m=0}^{n-1} D_{\beta,x}^+ D_x^- v_j^m \Big|^2 \\
&\leq C \Delta x^{2\beta},
\end{aligned}$$

where $\beta := \min\{\alpha, \gamma\}$ and we have assumed that the approximation of the initial data is of order Δx^β in L^2 and used (2.56) and (2.57). Hence we have that $p_x = \lim_{\Delta x \to 0} D_x^c p_{\Delta x} = \lim_{\Delta x \to 0} v_{\Delta x}/c_{\Delta x} = v/c$ and the limit p is the unique weak solution of the wave equation (1.1). Moreover, we have that $p_{\Delta x} \to p$ in $H^1([0,T] \times D) \cap C([0,T]; H^1(D))$ at a rate of at least $\Delta x^{\min\{\beta, (\alpha\gamma)/(2(\alpha\gamma+1-\gamma))}$, by Theorem 2.1.

2.2.2. *Numerical examples.* Next, we shall compare the above derived convergence rates to the ones in practice. To this end, we implement the finite difference scheme (2.6) and test it on a set of numerical test cases. For all the test case, we use the interval $D = [0, 2]$ as the computational domain and use periodic boundary conditions.

For the material coefficient c, we choose a sample (single realization) of a log-normally distributed random field, which was generated using a spectral FFT method [2, 15, 16, 14] from a given covariance operator $\hat{c}$ which we assume to be log-normal, so that the covariance operator completely determines the law of $\hat{c}$. It is easy to check that this coefficient c is uniformly positive, bounded from above and Hölder continuous with exponent $1/2$. See Figure 2 or an illustration of the coefficient. We compute approximations at time $T = 2$ and test the scheme on this set up with three different choices of initial data (with varying regularity),

$$p_{0,1}(x) = 1, \qquad u_{0,1}(x) = \sin(\pi x), \tag{2.58a}$$

$$p_{0,2}(x) = \begin{cases} x, & \text{if } x \leq 1, \\ 2 - x, & \text{if } x > 1, \end{cases} \qquad u_{0,2}(x) = \begin{cases} 1, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1, \end{cases} \tag{2.58b}$$

$$p_{0,3}(x) = 1, \qquad u_{0,3}(x) = \begin{cases} 1 + c(x), & \text{if } x \leq 1, \\ c(x), & \text{if } x > 1. \end{cases} \tag{2.58c}$$

We notice that according to Lemma 2.1, the moduli of continuity of the variables u and v is at least $\gamma = 1$ for the initial data (2.58a) whereas it is at least $\gamma = 1/2$ for initial data (2.58b) and (2.58c) In order to test the convergence, we have computed reference approximations on a grid

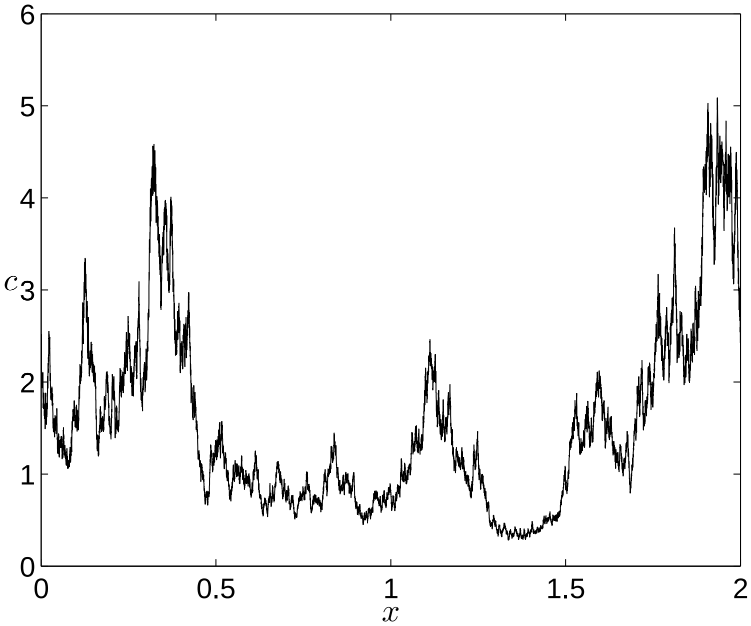

FIGURE 2. The coefficient c used for the numerical experiments for the 1d wave equation (2.2).

with $N_x = 2^{14}$ gridpoints. We have plotted the reference solutions to initial data (2.58a) and (2.58b) in Figure 3. For the approximation of the rate of convergence, we have used,

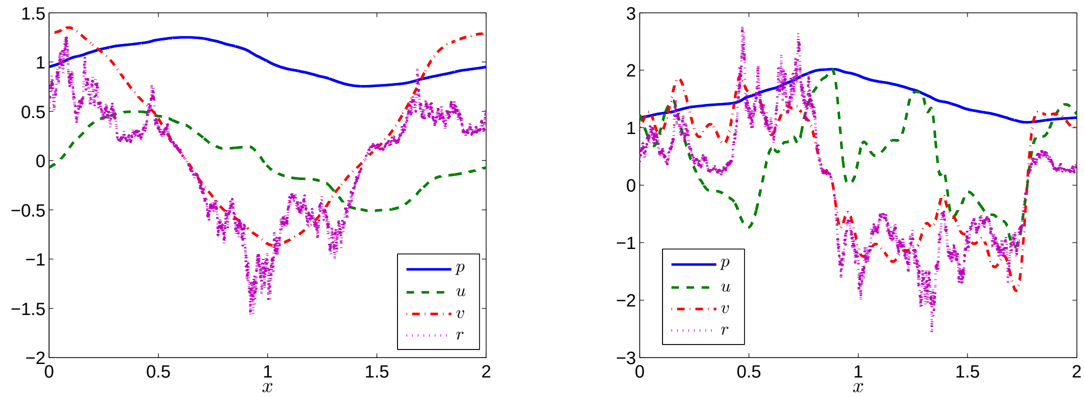

FIGURE 3. Left: Approximation of (2.2), (2.58a) at time $T = 2$ on a mesh with 2^{14} grid cells. Right: Approximation of (2.2), (2.58b) at time $T = 2$ on a mesh with 2^{14} grid cells.

$$r^2 = \frac{1}{N_{\text{exp}} - 1} \sum_{k=1}^{N_{\text{exp}}-1} \frac{\log \mathcal{E}^2_{\Delta x_k} - \log \mathcal{E}^2_{\Delta x_{k-1}}}{\log 2}, \tag{2.59}$$

where $\Delta x_k = 2^k \Delta x_0$, N_{exp} is the number of experiments and $\mathcal{E}^2_{\Delta x_k}$, the relative distance of the approximation with gridsize Δx_k to the reference solution in the discrete L^2-norm, that is,

$$\mathcal{E}^1_{\Delta x_k} = 100 \times \frac{\sqrt{\sum_{j=1}^{N_x} |\sigma_{\Delta x_k}(T, x_j) - \sigma_{\Delta x_{\text{ref}}}(T, x_j)|^2}}{\sqrt{\sum_{j=1}^{N_x} |\sigma_{\Delta x_{\text{ref}}}(T, x_j)|^2}}, \tag{2.60}$$

where $\sigma \in \{p, u, v, r\}$. We have used $\Delta x_0 = 1/32$ and $N_{\text{exp}} = 6$.

For the initial data (2.58a), we have observed a rate of ≈ 0.8 for the variables u and v, a rate of ≈ 0.7 for r and a rate of ≈ 0.95 for the variable p.

For the initial data (2.58b), we have observed a rate of ≈ 0.3 for the variables u, v and r and ≈ 0.75 for the variable p.

For the third set of initial data, (2.58c), we have observed a rate of ≈ 0.25 for the variables u, v and r and ≈ 0.65 for the variable p.

We note that these rates are better than the ones predicted by Theorem 2.1. However, instead of minimizing the expression (2.50) for a general set of parameters α, γ and Δx, one can minimize this expression directly for the given parameters α and γ and Δx_k, $k = 0, \ldots, 5$, and compute the approximate rate numerically from this expression. This program yields a rate of 0.74 (for the variables u, v) for the initial data (2.58a), and rates of approximately 0.37 for initial data (2.58b) and (2.58c). Note that these rates are very close to the experimentally observed rates indicating the sharpness of our method.

3. Two-dimensional version of (1.2)

Rates of convergence for finite difference approximations of the multi-dimensional wave equation (1.1) with Hölder continuous coefficients can be obtained in a fairly analogous manner as in the one-dimensional case discussed in the previous section. We illustrate this by considering the wave equation (1.2) in two space dimensions:

$$\begin{aligned} \partial_t u(t,x,y) - \partial_x(c(x,y) r_1(t,x,y)) - \partial_y(c(x,y) r_2(t,x,y)) &= 0, \\ \partial_t r_1(t,x,y) - \partial_x u(t,x,y) &= 0, \quad (t,x,y) \in D_T, \\ \partial_t r_2(t,x,y) - \partial_y u(t,x,y) &= 0, \end{aligned} \tag{3.1}$$

$D = [d_L^1, d_R^1] \times [d_L^2, d_R^2]$, $d_L^i < d_R^i \in [-\infty, \infty]$, $i = 1, 2$ and with periodic or Dirichlet boundary conditions extended by zero outside the domain. As before, we define the variables $v(t,x,y) := c(x,y) r_1(t,x,y)$ and $w(t,x,y) := c(x,y) r_2(t,x,y)$ to obtain the system,

$$\begin{aligned} u_t - v_x - w_y &= 0, \\ v_t - c\, u_x &= 0, \quad (t,x,y) \in D_T, \\ w_t - c\, u_y &= 0, \end{aligned} \tag{3.2}$$

As in the one-dimensional case, we find that system (3.2) is equipped with the L^2-entropy/entropy-flux pair,

$$\eta(u,v,w,c) := \frac{u^2}{2} + \frac{v^2}{2c} + \frac{w^2}{2c}, \quad q(u,v) := \begin{pmatrix} -uv \\ -uw \end{pmatrix} \tag{3.3}$$

so that formally,

$$\eta(u-k, v-\ell, w-m, c)_t + \operatorname{div} q(u-k, v-\ell, w-m) = 0. \tag{3.4}$$

for a solution (u,v,w) of (3.2), which implies that if the L^2-norms of u, v and w are bounded initially, they will be bounded for any time.

3.0.3. *Numerical approximation of* (3.2) *by a finite difference scheme.* To approximate the solution of (3.2), we let $\Delta x > 0$ and discretize the spatial domain by a grid with gridpoints $(x_{i+1/2}, y_{j+1/2}) = (i\Delta x, j\Delta x)$, for $i, j \in \mathbb{Z}$ for which $(x_{i+1/2}, y_{j+1/2}) \in D$. We denote the grid cells $\mathcal{C}_{ij} := [x_{i-1/2}, x_{i+1/2}) \times [y_{j-1/2}, y_{j+1/2})$. Furthermore, fix $\kappa > 0$ and let $\Delta t = \theta \Delta x$, where θ satisfies the CFL-condition

$$\theta < \min\left\{\frac{1}{\kappa^2 + \bar{c}}, \frac{1}{1 + 4\kappa^2 \bar{c}}\right\} \frac{\kappa}{2}. \tag{3.5}$$

Set $t^n := n\Delta t$, $n = 0, 1, 2, \ldots$, and define N_T such that $t^{N_T} = T$. Moreover, define the averaged quantities,

$$c_{ij} = \frac{1}{\Delta x^2} \int_{\mathcal{C}_{ij}} c(x, y)\, dx\, dy, \qquad i, j \in \mathbb{Z}, \tag{3.6}$$

and

$$\begin{aligned} u_{ij}^0 &= \frac{1}{\Delta x^2} \int_{\mathcal{C}_{ij}} u_0(x, y)\, dx\, dy, \\ v_{ij}^0 &= \frac{1}{\Delta x^2} \int_{\mathcal{C}_{ij}} v_0(x, y)\, dx\, dy, \qquad i, j \in \mathbb{Z}, \\ w_{ij}^0 &= \frac{1}{\Delta x^2} \int_{\mathcal{C}_{ij}} w_0(x, y)\, dx\, dy. \end{aligned} \tag{3.7}$$

Now we define approximations u_{ij}^n, v_{ij}^n and w_{ij}^n via the following numerical scheme,

$$D_t^+ u_{ij}^n = D_x^- v_{ij}^n + D_y^+ w_{ij}^n + \Delta x\, \kappa\, D_x^+ D_x^- u_{ij}^n + \Delta x\, \kappa\, D_y^+ D_y^- u_{ij}^n, \tag{3.8a}$$

$$D_t^+ \frac{v_{ij}^n}{c_{ij}} = D_x^+ u_{ij}^n + \Delta x\, \kappa\, D_x^+ D_x^- v_{ij}^n + \Delta x\, \kappa\, D_y^+ D_x^+ w_{ij}^n, \quad i, j \in \mathbb{Z},\ n = 0, \ldots, N, \tag{3.8b}$$

$$D_t^+ \frac{w_{ij}^n}{c_{ij}} = D_y^- u_{ij}^n + \Delta x\, \kappa\, D_x^- D_y^- v_{ij}^n + \Delta x\, \kappa\, D_y^+ D_y^- w_{ij}^n, \tag{3.8c}$$

for $\kappa > 0$, D_t^+ as defined in (2.5) and

$$D_x^\pm \sigma_{ij}^n := \pm \frac{1}{\Delta x}(\sigma_{i\pm1,j}^n - \sigma_{ij}^n), \quad D_y^\pm \sigma_{ij}^n := \pm \frac{1}{\Delta x}(\sigma_{i,j\pm1}^n - \sigma_{ij}^n), \tag{3.9}$$

for a quantity σ_{ij}^n, $i, j \in \mathbb{Z}$, $n = 0, \ldots, N_T$ defined on the grid. In addition, we denote the discrete entropy function and flux by

$$\eta_{ij}^n := \frac{\left|u_{ij}^n - k\right|^2}{2} + \frac{\left|v_{ij}^n - \ell\right|^2}{2c_{ij}} + \frac{\left|w_{ij}^n - m\right|^2}{2c_{ij}}, \quad q_{ij}^n := \begin{pmatrix} -(u_{ij}^n - k)(v_{ij}^n - \ell) \\ -(u_{ij}^n - k)(w_{ij}^n - m) \end{pmatrix}. \tag{3.10}$$

The scheme (3.8) satisfies the following properties:

Lemma 3.1. *Assume $c \in C^{0,\alpha}(D)$ and $u_0, v_0, w_0 \in L^2(D)$. Then the numerical approximations u_{ij}^n, v_{ij}^n and w_{ij}^n defined by (3.8), (3.6) and (3.7) have the following properties:*

(i) *Discrete entropy inequality:*

$$\begin{aligned} &D_t^+ \eta_{ij}^n - D_x^- \left((u_{ij}^n - k)(v_{ij}^n - \ell)\right) - D_y^+ \left((u_{ij}^n - k)(w_{ij}^n - m)\right) \\ &\quad \leq \Delta x \Big[D_x^- \left((v_{ij}^n - \ell) D_x^+ (u_{ij}^n - k)\right) - D_y^+ \left((w_{ij}^n - m) D_y^- (u_{ij}^n - k)\right) \\ &\qquad + \frac{\kappa}{2} \Big\{ D_x^+ \left((v_{ij} + v_{i-1,j} - 2\ell)\left(D_y^+ (w_{ij} - m) + D_x^- (v_{ij} - \ell)\right)\right) \\ &\qquad\quad + D_y^- \left((w_{ij} + w_{i,j+1} - 2m)\left(D_y^+ (w_{ij} - m) + D_x^- (v_{ij} - \ell)\right)\right) \Big\} \Big] \\ &\qquad + \frac{\kappa \Delta x}{2} \hat{\Delta}(u_{ij}^n - k)^2 - \frac{\kappa \Delta x^2}{4} \Big[D_x^- (|D_x^+ (v_{ij} - \ell)|^2) + D_y^- (|D_x^- (v_{ij} - \ell)|^2) \\ &\qquad\qquad - D_x^+ (|D_y^+ (w_{ij} - m)|^2) - D_y^+ (|D_y^- (w_{ij} - m)|^2) \Big], \end{aligned} \tag{3.11}$$

for $i, j \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$

(ii) *Bounds on the discrete L^2-norms:*

$$\Delta x^2 \sum_{ij} (u_{ij}^n)^2 + \frac{1}{c_{ij}}(v_{ij}^n)^2 + \frac{1}{c_{ij}}(w_{ij}^n)^2 \leq \Delta x^2 \sum_{ij} (u_{ij}^0)^2 + \frac{1}{c_{ij}}(v_{ij}^0)^2 + \frac{1}{c_{ij}}(w_{ij}^0)^2 \tag{3.12}$$
$$\leq \|u_0\|_{L^2}^2 + \left\|c^{-1/2} v_0\right\|_{L^2}^2 + \left\|c^{-1/2} w_0\right\|_{L^2}^2 .$$

for $n = 0, \ldots, N_T$.

(iii) Vorticity preservation*:*

$$D_{\gamma,y}^- \left(\frac{v_{ij}^n}{c_{ij}}\right) - D_{\gamma,x}^+ \left(\frac{w_{ij}^n}{c_{ij}}\right) = D_{\gamma,y}^- \left(\frac{v_{ij}^0}{c_{ij}}\right) - D_{\gamma,x}^+ \left(\frac{w_{ij}^0}{c_{ij}}\right), \tag{3.13}$$

for $i, j \in \mathbb{Z},$, $n = 0, \ldots, N_T$ *and* $\gamma \in (0, 1]$.

(iv) *If we assume in addition that the initial data* u_0, v_0 *and* w_0 *have moduli of continuity in* $L^2(D)$,

$$\nu_x^2(u_0,\sigma) \leq C\,\sigma^{2\gamma}, \quad \nu_x^2(v_0,\sigma) \leq C\,\sigma^{2\gamma}, \quad \nu_x^2(w_0,\sigma) \leq C\,\sigma^{2\gamma}$$
$$\nu_y^2(u_0,\sigma) \leq C\,\sigma^{2\gamma}, \quad \nu_y^2(v_0,\sigma) \leq C\,\sigma^{2\gamma}, \quad \nu_y^2(w_0,\sigma) \leq C\,\sigma^{2\gamma},$$

for some $\gamma > 0$, *the approximations satisfy,*

$$\Delta x^2 \sum_{ij} |D_{\gamma,t}^+ u_{ij}^n|^2 + \frac{1}{c_{ij}}|D_{\gamma,t}^+ v_{ij}^n|^2 + \frac{1}{c_{ij}}|D_{\gamma,t}^+ w_{ij}^n|^2 \leq C_1, \tag{3.14a}$$

$$\Delta x^2 \sum_{ij} |D_{\gamma,x}^+ u_{ij}^n|^2 + |D_{\gamma,y}^- u_{ij}^n|^2 + |D_{\gamma,x}^- v_{ij}^n + D_{\gamma,y}^+ w_{ij}^n|^2 \leq C_1, \tag{3.14b}$$

where C_1 *is a constant depending on* c *and the initial data* u_0, v_0 *and* w_0, *and for* $\beta \leq \min\{\alpha, \gamma\}$,

$$\Delta x^2 \sum_{ij} |D_{\beta,x}^- v_{ij}^n|^2 + |D_{\beta,y}^- v_{ij}^n|^2 + |D_{\beta,x}^+ w_{ij}^n|^2 + |D_{\beta,y}^+ w_{ij}^n|^2 \leq C_2, \tag{3.15}$$

with C_2 *is a constant depending on the Hölder norm of* c *and the initial data* u_0, v_0 *and* w_0, *and where we have denoted*

$$D_{\gamma,x}^\pm \sigma_{ij}^n = \mp \frac{\sigma_{ij}^n - \sigma_{i\pm1,j}^n}{\Delta x^\gamma}, \quad D_{\gamma,y}^\pm \sigma_{ij}^n = \mp \frac{\sigma_{ij}^n - \sigma_{i,j\pm1}^n}{\Delta x^\gamma}, \tag{3.16}$$

for a quantity σ_{ij}^n *defined on our grid.*

Proof. (i) By linearity, it is enough to show this for $k = \ell = m = 0$. For ease of notation we drop the superscript n when calculating the cell entropy equation. We multiply (3.8a) with u_{ij}, (3.8b) with v_{ij} and (3.8c) with w_{ij} to get

$$\frac{1}{2} D_t^+ u_{ij}^2 - \frac{\Delta t}{2}\left(D_t^+ u_{ij}\right)^2 = u_{ij} D_x^- v_{ij} + u_{ij} D_y^+ w_{ij} + \frac{\kappa \Delta x}{2} \hat{\Delta} u_{ij}^2$$
$$- \frac{\kappa \Delta x}{2}\left(\left(D_x^- u_{ij}\right)^2 + \left(D_x^+ u_{ij}\right)^2 + \left(D_y^- u_{ij}\right)^2 + \left(D_y^+ u_{ij}\right)^2\right),$$
$$\frac{1}{2c_{ij}} D_t^+ v_{ij}^2 - \frac{\Delta t}{2c_{ij}}\left(D_t^+ v_{ij}\right)^2 = v_{ij} D_x^+ u_{ij} + \frac{\kappa \Delta x}{2} D_x^+ D_x^- v_{ij}^2 - \frac{\kappa \Delta x}{2}\left(\left(D_x^- v_{ij}\right)^2 + \left(D_x^+ v_{ij}\right)^2\right)$$
$$+ \kappa \Delta x v_{ij} D_y^+ D_x^+ w_{ij},$$
$$\frac{1}{2c_{ij}} D_t^+ w_{ij}^2 - \frac{\Delta t}{2c_{ij}}\left(D_t^+ w_{ij}\right)^2 = w_{ij} D_y^- u_{ij} + \frac{\kappa \Delta x}{2} D_y^+ D_y^- w_{ij}^2 - \frac{\kappa \Delta x}{2}\left(\left(D_y^- w_{ij}\right)^2 + \left(D_y^+ w_{ij}\right)^2\right)$$
$$+ \kappa \Delta x w_{ij} D_y^- D_x^- v_{ij},$$

where we have used the shorthand notation $\hat{\Delta} = D_x^+ D_x^- + D_y^+ D_y^-$. Adding these three equations we get

$$D_t^+ \eta_{ij} = \underbrace{u_{ij} D_x^- v_{ij} + v_{ij} D_x^+ u_{ij} + u_{ij} D_y^+ w_{ij} + w_{ij} D_y^- u_{ij}}_{\mathbf{a}}$$

$$
\begin{aligned}
&+\frac{\kappa\Delta x}{2}\underbrace{\left(\hat{\Delta}u_{ij}^2 + D_x^+D_x^-v_{ij}^2 + D_y^+D_y^-w_{ij}^2\right)}_{\mathbf{b}}\\
&-\frac{\kappa\Delta x}{2}\underbrace{\left[\frac{1}{2}\left(D_x^+v_{ij}+D_y^+w_{i+1,j}\right)^2 + \left(D_x^-v_{ij}+D_y^+w_{ij}\right)^2 + \frac{1}{2}\left(D_x^-v_{i,j-1}+D_y^-w_{ij}\right)^2\left|\bar{D}_{x,y}u_{ij}\right|^2\right]}_{\mathbf{c}}\\
&+\kappa\Delta x\underbrace{\left(v_{ij}D_x^+D_y^+w_{ij} + w_{ij}D_x^-D_y^-v_{ij} + D_y^+w_{ij}D_x^-v_{ij} + \frac{1}{2}D_x^-v_{i,j-1}D_y^-w_{ij} + \frac{1}{2}D_x^+v_{ij}D_y^+w_{i+1,j}\right)}_{\mathbf{d}}\\
&+\frac{\Delta t}{2}\underbrace{\left(\left(D_t^+u_{ij}\right)^2 + \frac{1}{c_{ij}}\left(\left(D_t^+v_{ij}\right)^2 + \left(D_t^+w_{ij}\right)^2\right)\right)}_{\mathbf{e}}\\
&-\frac{\kappa\Delta x}{4}\underbrace{\left(|D_x^+v_{ij}|^2 - |D_x^-v_{i,j-1}|^2 + |D_y^-w_{ij}|^2 - |D_y^+w_{i+1,j}|^2\right)}_{\mathbf{f}},
\end{aligned}
$$

where

$$\left|\bar{D}_{x,y}u_{ij}\right|^2 = \left(D_x^-u_{ij}\right)^2 + \left(D_x^+u_{ij}\right)^2 + \left(D_y^-u_{ij}\right)^2 + \left(D_y^+u_{ij}\right)^2.$$

To proceed further, we utilize the identities

$$
\begin{aligned}
\alpha_i\left(D^-\beta_i\right) + \left(D^+\alpha_i\right)\beta_i &= D^-\left(\alpha_{i+1}\beta_i\right)\\
&= D^+\left(\alpha_i\beta_i\right) - \Delta x D^+\left(\alpha_i D^-\beta_i\right)\\
&= D^-(\alpha_i\beta_i) + \Delta x D^-(\beta_i D^+\alpha_i).
\end{aligned}
$$

Using this

$$\mathbf{a} = D_x^-\left(u_{ij}v_{ij}\right) + D_y^+\left(u_{ij}w_{ij}\right) + \Delta x\underbrace{\left(D_x^-\left(v_{ij}D_x^+u_{ij}\right) - D_y^+\left(w_{ij}D_y^-u_{ij}\right)\right)}_{\mathbf{a}_1},$$

and for $\mathbf{d}$ we find

$$\mathbf{d} = \frac{1}{2}\left(D_x^+(v_{ij}D_y^+w_{ij}) + D_x^-(v_{ij}D_y^+w_{i+1,j}) + D_y^+(w_{ij}D_x^-v_{i,j-1}) + D_y^-(w_{ij}D_x^-v_{ij})\right).$$

We estimate the term $\mathbf{e}$ as,

$$
\begin{aligned}
\mathbf{e} &\le 2\left(D_x^-v_{ij}+D_y^+w_{ij}\right)^2 + 2\kappa^2\Delta x^2\left(\hat{\Delta}u_{ij}\right)^2\\
&\quad + 2\bar{c}\Big(\left(D_x^+u_{ij}\right)^2 + \left(D_y^-u_{ij}\right)^2 + \kappa^2\Delta x^2\left(D_x^+\left(D_x^-v_{ij}+D_y^+w_{ij}\right)\right)^2\\
&\qquad + \kappa^2\Delta x^2\left(D_y^-\left(D_x^-v_{ij}+D_y^+w_{ij}\right)\right)^2\Big)\\
&\le 2\left(D_x^-v_{ij}+D_y^+w_{ij}\right)^2 + 2\kappa^2\left|\bar{D}_{x,y}u_{ij}\right|^2\\
&\quad + 2\bar{c}\Big(\left|\bar{D}_{x,y}u_{ij}\right|^2 + 2\kappa^2\left(\left(D_x^-v_{ij}+D_y^+w_{ij}\right)^2 + \left(D_x^-v_{i+1,j}+D_y^+w_{i+1,j}\right)^2\right)\\
&\qquad + 2\kappa^2\left(\left(D_x^-v_{ij}+D_y^+w_{ij}\right)^2 + \left(D_x^-v_{i,j-1}+D_y^+w_{i,j-1}\right)^2\right)\Big)\\
&= 2(\kappa^2+\bar{c})\left|\bar{D}_{x,y}u_{ij}\right|^2 + 2\left(1+4\bar{c}\kappa^2\right)\left(D_x^-v_{ij}+D_y^+w_{ij}\right)^2\\
&\quad + 4\bar{c}\kappa^2\left(\left(D_x^-v_{i+1,j}+D_y^+w_{i+1,j}\right)^2 + \left(D_x^-v_{i,j-1}+D_y^+w_{i,j-1}\right)^2\right)
\end{aligned}
$$

Using this inequality

$$
\begin{aligned}
\frac{\Delta t}{2}\mathbf{e} - \frac{\kappa\Delta x}{2}\mathbf{c} &\le \left(\left(\kappa^2+\bar{c}\right)\Delta t - \frac{\kappa\Delta x}{2}\right)\left|\bar{D}_{x,y}u_{ij}\right|^2\\
&\quad + \left(\left(1+4\kappa^2\bar{c}\right)\Delta t - \frac{\kappa\Delta x}{2}\right)\left(D_x^-v_{ij}+D_y^+w_{ij}\right)^2\\
&\quad + \left(2\kappa^2\bar{c}\Delta t - \frac{\kappa\Delta x}{4}\right)\left(\left(D_x^-v_{i+1,j}+D_y^+w_{i+1,j}\right)^2 + \left(D_x^-v_{i,j-1}+D_y^+w_{i,j-1}\right)^2\right)
\end{aligned}
$$

if the CFL-condition (3.5) holds. The term $\mathbf{f}$ can be rewritten as

$$\mathbf{f} = \Delta x \left(D_x^-(|D_x^+ v_{ij}|^2) + D_y^-(|D_x^- v_{ij}|^2) - D_x^+(|D_y^+ w_{ij}|^2) - D_y^+(|D_y^- w_{ij}|^2) \right)$$

Thus

$$D_t^+ \eta_{ij} - D_x^- (u_{ij} v_{ij}) - D_y^+ (u_{ij} w_{ij}) \leq \Delta x \mathbf{a}_1 + \frac{\kappa \Delta x}{2} \mathbf{b} + \kappa \Delta x \mathbf{d} - \frac{\kappa \Delta x}{4} \mathbf{f} \tag{3.17}$$

which is (3.11).

(ii) This follows from (i) by setting $k = \ell = m = 0$ in (3.11) and summing over all $i, j \in \mathbb{Z}$.

(iii) We apply the operator $D_{\gamma,y}^-$ to both sides of the evolution equation for v_{ij}^n, (3.8b), and $D_{\gamma,x}^+$ to the expressions in the evolution equation for w_{ij}^n, (3.8c). Subtracting the resulting equations, we obtain

$$\begin{aligned} D_t^+ &\left(D_{\gamma,y}^- \left(\frac{v_{ij}^n}{c_{ij}} \right) - D_{\gamma,x}^+ \left(\frac{w_{ij}^n}{c_{ij}} \right) \right) \\ &= D_{\gamma,y}^- D_x^+ u_{ij}^n - D_{\gamma,x}^+ D_y^- u_{ij}^n \\ &\quad + \Delta x \kappa \left[D_{\gamma,y}^- D_x^+ D_x^- v_{ij}^n + D_{\gamma,y}^- D_y^+ D_x^+ w_{ij}^n - D_{\gamma,x}^+ D_x^- D_y^- v_{ij}^n - D_{\gamma,x}^+ D_y^+ D_y^- w_{ij}^n \right] \\ &= 0, \end{aligned}$$

thus (iii) follows by induction over n.

(iv) As in the one-dimensional case, we observe that the differences $D_{\gamma,t}^+ u_{ij}^n$, $D_{\gamma,t}^+ v_{ij}^n$ and $D_{\gamma,t}^+ w_{ij}^n$ satisfy the same equations (3.8) as u_{ij}^n, v_{ij}^n and w_{ij}^n, so that (3.12) holds,

$$\begin{aligned} \Delta x^2 \sum_{ij} \left| D_{\gamma,t}^+ u_{ij}^n \right|^2 + \frac{1}{c_{ij}} \left| D_{\gamma,t}^+ v_{ij}^n \right|^2 + \frac{1}{c_{ij}} \left| D_{\gamma,t}^+ w_{ij}^n \right|^2 & \\ \leq \Delta x^2 \sum_{ij} \left| D_{\gamma,t}^+ u_{ij}^0 \right|^2 + \frac{1}{c_{ij}} \left| D_{\gamma,t}^+ v_{ij}^0 \right|^2 + \frac{1}{c_{ij}} \left| D_{\gamma,t}^+ w_{ij}^0 \right|^2 &, \end{aligned} \tag{3.18}$$

We take the squares of equations (3.8) and sum over all i, j to obtain,

$$\begin{aligned} &\sum_{ij} \left| D_t^+ u_{ij}^n \right|^2 + \frac{1}{c_{ij}^2} \left| D_t^+ v_j^n \right|^2 + \frac{1}{c_{ij}^2} \left| D_t^+ w_j^n \right|^2 \\ &= 2 \sum_{ij} \Big(\left| D_x^- v_{ij}^n + D_y^- w_{ij}^n \right|^2 + \left| D_x^- u_{ij}^n \right|^2 + \left| D_y^- u_{ij}^n \right|^2 \\ &\qquad + \Delta x^2 \kappa^2 \left(\left| \hat{\Delta} u_{ij}^n \right|^2 + \left| D_x^+ D_x^- v_{ij}^n + D_y^+ D_x^+ w_{ij}^n \right|^2 + \left| D_y^- D_x^- v_{ij}^n + D_y^- D_x^- w_{ij}^n \right|^2 \right) \Big), \\ &\leq 2 \sum_{ij} \left((1 + 8\kappa^2) \left| D_x^- v_{ij}^n + D_y^+ w_{ij}^n \right|^2 + (1 + 4\kappa^2) \left(\left| D_x^- u_{ij}^n \right|^2 + \left| D_x^+ u_{ij}^n \right|^2 \right) \right) \end{aligned} \tag{3.19}$$

Combining (3.18), (3.19) for $n = 0$, the CFL-condition (3.5) and the assumption that the initial data has a modulus of continuity, we obtain the inequality (3.14a):

$$\begin{aligned} \Delta x^2 &\sum_{ij} \left| D_{\gamma,t}^+ u_{ij}^n \right|^2 + \frac{1}{c_{ij}} \left| D_{\gamma,t}^+ v_{ij}^n \right|^2 + \frac{1}{c_{ij}} \left| D_{\gamma,t}^+ w_{ij}^n \right|^2 \\ &\leq \Delta x^2 \sum_{ij} \left| D_{\gamma,t}^+ u_{ij}^0 \right|^2 + \frac{1}{c_{ij}} \left| D_{\gamma,t}^+ v_{ij}^0 \right|^2 + \frac{1}{c_{ij}} \left| D_{\gamma,t}^+ w_{ij}^0 \right|^2 \\ &\leq \Delta x^2 \max\left\{1, \bar{c}\right\} \sum_{ij} \left| D_{\gamma,t}^+ u_{ij}^0 \right|^2 + \frac{1}{c_{ij}^2} \left| D_{\gamma,t}^+ v_j^0 \right|^2 + \frac{1}{c_{ij}^2} \left| D_{\gamma,t}^+ w_j^0 \right|^2 \\ &\leq \Delta x^2 \max\left\{1, \bar{c}\right\} \theta^{2-2\gamma} 2 \sum_{ij} \Big((1 + 8\kappa^2) \left| D_{\gamma,x}^- v_{ij}^0 + D_{\gamma,y}^+ w_{ij}^0 \right|^2 \\ &\qquad\qquad + (1 + 4\kappa^2) \left(\left| D_{\gamma,x}^+ u_{ij}^0 \right|^2 + \left| D_{\gamma,y}^- u_{ij}^0 \right|^2 \right) \Big) \\ &\leq C(u_0, v_0, w_0). \end{aligned}$$

By (3.14a) and (3.19) we furthermore obtain (3.14b). We observe that in contrast to the one-dimensional case, we do not obtain a modulus of continuity in space for the variables v and w in this way but only the bound

$$\Delta x^2 \sum_{ij} \left| D^-_{\gamma,x} v^n_{ij} + D^+_{\gamma,y} w^n_{ij} \right|^2 \leq C(u_0, v_0, w_0).$$

However, by the "vorticity" conservation (3.13) and the bound on the discrete L^2-norm, (3.12) we have

$$\begin{aligned}
\Delta x^2 \sum_{ij} \left| D^-_{\beta,y} v^n_{ij} - D^+_{\beta,x} w^n_{ij} \right|^2 &\leq 2\bar{c}\Delta x^2 \sum_{ij} \big| D^-_{\beta,y}\big(\frac{v^n_{ij}}{c_{ij}}\big) - D^+_{\beta,x}\big(\frac{w^n_{ij}}{c_{ij}}\big)\big|^2 \\
&\quad + \frac{4\Delta x^2}{\underline{c}^2} \|c\|^2_{C^{0,\beta}} \sum_{ij} \left(\left|v^n_{ij}\right|^2 + \left|w^n_{ij}\right|^2 \right) \\
&\leq 2\bar{c}\Delta x^2 \sum_{ij} \big| D^-_{\beta,y}\big(\frac{v^0_{ij}}{c_{ij}}\big) - D^+_{\beta,x}\big(\frac{w^0_{ij}}{c_{ij}}\big)\big|^2 \\
&\quad + \frac{4}{\underline{c}^2} \|c\|^2_{C^{0,\beta}} \left(\|v_0\|^2_{L^2} + \|w_0\|^2_{L^2} \right) \\
&\leq C(v_0, w_0, c),
\end{aligned}$$

for $\beta \leq \min\{\gamma, \alpha\}$. Thus, using summation by parts, we obtain the bound (3.15):

$$\begin{aligned}
\Delta x^2 \sum_{ij} &\left| D^-_{\beta,x} v^n_{ij} \right|^2 + \left| D^-_{\beta,y} v^n_{ij} \right|^2 + \left| D^+_{\beta,x} w^n_{ij} \right|^2 + \left| D^+_{\beta,y} w^n_{ij} \right|^2 \\
&= \Delta x^2 \sum_{ij} \left| D^-_{\beta,x} v^n_{ij} + D^+_{\beta,y} w^n_{ij} \right|^2 + \left| D^-_{\beta,y} v^n_{ij} - D^+_{\beta,x} w^n_{ij} \right|^2 \\
&\leq C(u_0, v_0, w_0, c).
\end{aligned}$$

□

We define the piecewise constant functions,

$$u_{\Delta x}(t, x, y) = u^n_{ij}, \quad (t, x, y) \in [t^n, t^{n+1}) \times \mathcal{C}_{ij}, \tag{3.20a}$$

$$v_{\Delta x}(t, x, y) = v^n_{ij}, \quad (t, x, y) \in [t^n, t^{n+1}) \times \mathcal{C}_{ij}, \tag{3.20b}$$

$$w_{\Delta x}(t, x, y) = w^n_{ij}, \quad (t, x, y) \in [t^n, t^{n+1}) \times \mathcal{C}_{ij}, \tag{3.20c}$$

$$c_{\Delta x}(x, y) = c_{ij}, \quad (x, y) \in \mathcal{C}_{ij}, \tag{3.20d}$$

so that Lemma 3.1 combined with Kolmogorov's compactness theorem imply that a subsequence of $(u_{\Delta x}, v_{\Delta x}, w_{\Delta x})_{\Delta x > 0}$ converges in $C([0, T]; L^2(D))$, as $\Delta x \to 0$, to a unique limit $(u, v, w) \in C([0, T]; L^2(D))$ which is a weak solution of (3.2) and satisfies an entropy inequality. Moreover, (u, v, w) have the same moduli of continuity as the discrete approximations and in particular,

$$u, v, w \in L^\infty([0, T]; H^s(D)) \cap C^{0,\min\{\alpha,\gamma\}}([0, T]; L^2(D)) \quad 0 < s \leq \min\{\gamma, \alpha\}. \tag{3.21}$$

3.0.4. *Convergence rate.* For $\phi \in C^2_0((0, T) \times D)$ define

$$\begin{aligned}
\Lambda_T(u, v, w, k, \ell, m, \phi) := \int_{D_T} \Bigg\{ &\left(\frac{(u - k)^2}{2} + \frac{(v - \ell)^2}{2c} + \frac{(w - m)^2}{2c} \right) \phi_t \\
&- (u - k)(v - \ell)\phi_x - (u - k)(w - m)\phi_y \Bigg\} \, dx dy dt
\end{aligned} \tag{3.22}$$

Furthermore, we define the test function $\Omega \in C^\infty_0((0, T) \times D)$ by

$$\Omega(t, s, x_1, y_1, x_2, y_2) = \psi^\mu(t)\omega_{\epsilon_0}(t - s)\omega_\epsilon(x_1 - y_1)\omega_\epsilon(x_2 - y_2), \tag{3.23}$$

where ω and ψ^μ are the functions introduced in the proof of theorem 2.1. Then we have

Lemma 3.2. *Let $c \in C^{0,\alpha}(D)$ satisfy $\bar{c} > c(x,y) > \underline{c}$ for all $(x,y) \in D$. Denote (u,v,w) the solution of (3.2) and $(u_{\Delta x}, v_{\Delta x}, w_{\Delta x})$ the numerical approximation computed by scheme (3.8) and defined in (3.20). Assume that the initial data u_0, v_0 and w_0 are in $L^2(D)$ have moduli of continuity*

$$\nu_x^2(u_0,\sigma) \le C\,\sigma^{2\gamma}, \quad \nu_x^2(v_0,\sigma) \le C\,\sigma^{2\gamma}, \quad \nu_x^2(w_0,\sigma) \le C\,\sigma^{2\gamma},$$
$$\nu_y^2(u_0,\sigma) \le C\,\sigma^{2\gamma}, \quad \nu_y^2(v_0,\sigma) \le C\,\sigma^{2\gamma}, \quad \nu_y^2(w_0,\sigma) \le C\,\sigma^{2\gamma}.$$

for some $\gamma \in (0,1]$. Then $((u_{\Delta x}, v_{\Delta x}, w_{\Delta x})(t,\cdot))$ converges to the solution $((u,v,w)(t,\cdot))$, $0 < t < T$, at (at least) the rate

$$\begin{aligned}(3.24)\quad \|(u-u_{\Delta x})(t,\cdot)\|_{L^2(D)} &+ \|(v-v_{\Delta x})(t,\cdot)/c\|_{L^2(D)} + \|(w-w_{\Delta x})(t,\cdot)/c\|_{L^2(D)} \\ &\le C\big(\|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/c\|_{L^2(D)} \\ &\qquad + \|(w_0 - w_{\Delta x}(0,\cdot))/c\|_{L^2(D)} + \Delta x^s\big),\end{aligned}$$

where C is a constant depending on $\underline{c}$, $\bar{c}$, $\|c\|_{C^{0,\alpha}}$ and T but not on Δx, and s is given by

$$(3.25)\qquad s = \begin{cases} \alpha\gamma^2/(\alpha\gamma+1), & \alpha \ge (2\gamma-1)/(\gamma(2-\gamma)) \\ \alpha\gamma/(\alpha\gamma+2-\alpha), & \alpha \le (2\gamma-1)/(\gamma(2-\gamma)). \end{cases}$$

Proof. We assume without loss of generality that $\Delta x \le \min\{\epsilon, \epsilon_0, \nu\}$. In the following we shall use that $u = u(t, x_1, y_1)$ and that $u_{\Delta x} = u_{\Delta x}(s, x_2, y_2)$.

We first note that, since (u,v,w) is a solution of (3.2),

$$\begin{aligned}(3.26)\quad \int_{D_T} \left(\frac{(u-u_{\Delta x})^2}{2} + \frac{(v-v_{\Delta x})^2}{2c} + \frac{(w-w_{\Delta x})^2}{2c} \right)\phi_t \\ - (u-u_{\Delta x})(v-v_{\Delta x})\phi_{x_1} - (u-u_{\Delta x})(w-w_{\Delta x})\phi_{y_1}\, dx_1 dy_1 dt \ge 0.\end{aligned}$$

for all $(s, x_2, y_2) \in D_T$. On the other hand, $(u_{\Delta x}, v_{\Delta x}, w_{\Delta x})$ satisfies the cell entropy inequality (3.11),

(3.27)
$$\begin{aligned}
&\int_{D_T} \left(\frac{(u_{\Delta x}-u)^2}{2} + \frac{(v_{\Delta x}-v)^2}{2c_{\Delta x}} + \frac{(w_{\Delta x}-w)^2}{2c_{\Delta x}} \right) D_s^- \phi \\
&\qquad - (u_{\Delta x}-u)(v_{\Delta x}-v) D_{x_2}^+ \phi - (u_{\Delta x}-u)(w_{\Delta x}-w) D_{y_2}^- \phi \, dx_2 dy_2 ds \\
&\ge \Delta x \int_{D_T} \left[(v_{\Delta x}-v) D_{x_2}^+ (u_{\Delta x}-u) D_{x_2}^+ \phi + (w_{\Delta x}-w) D_{y_2}^- (u_{\Delta x}-u) D_{y_2}^- \phi \right] dx_2 dy_2 ds \\
&\quad + \frac{\kappa \Delta x}{2} \int_{D_T} \left\{ (v_{\Delta x} + v_{\Delta x}(\cdot, x_2 - \Delta x, \cdot) - 2v) D_{x_2}^- \phi + (w_{\Delta x} + w_{\Delta x}(\cdot, y_2 + \Delta x, \cdot) - 2w) D_{y_2}^+ \phi \right\} \\
&\qquad\qquad \times \left(D_{y_2}^+ (w_{\Delta x} - w) + D_{x_2}^- (v_{\Delta x} - v) \right) dx_2 dy_2 ds \\
&\quad + \frac{\kappa \Delta x}{2} \int_{D_T} D_{x_2}^+ (u_{\Delta x} - u)^2 D_{x_2}^+ \phi + D_{y_2}^+ (u_{\Delta x} - u)^2 D_{y_2}^+ \phi \, dx_2 dy_2 ds \\
&\quad + \frac{\kappa \Delta x^2}{4} \int_{D_T} \Big\{ D_{x_2}^- \left(|D_{x_2}^+ (v_{\Delta x} - v)|^2 \right) + D_{y_2}^- \left(|D_{x_2}^- (v_{\Delta x} - v)|^2 \right) \\
&\qquad\qquad - D_{x_2}^+ \left(|D_{y_2}^+ (w_{\Delta x} - w)|^2 \right) - D_{y_2}^+ \left(|D_{y_2}^- (w_{\Delta x} - w)|^2 \right) \Big\} \phi \, dx_2 dy_2 ds,
\end{aligned}$$

for all (t, x_1, y_1). Adding (3.26) and (3.27), using Ω as test function and integrating over D_T, we obtain,

$$\underbrace{\int_{D_T^2} \left(\frac{(u_{\Delta x}-u)^2}{2} + \frac{(v_{\Delta x}-v)^2}{2c} + \frac{(w_{\Delta x}-w)^2}{2c} \right) (\Omega_t + D_s^- \Omega) \, d\underline{z}}_{A}$$

$$\underbrace{-\int_{D_T^2}(u_{\Delta x}-u)(v_{\Delta x}-v)(\Omega_{x_1}+D^+_{x_2}\Omega)+(u_{\Delta x}-u)(w_{\Delta x}-w)(\Omega_{y_1}+D^-_{y_2}\Omega)\,d\underline{z}}_{B}$$

$$\geq \underbrace{\int_{D_T^2}\left((v_{\Delta x}-v)^2+(w_{\Delta x}-w)^2\right)\left(\frac{1}{2c(x)}-\frac{1}{2c_{\Delta x}(y)}\right)D_s^-\Omega\,d\underline{z}}_{D}$$

$$+\underbrace{\Delta x\int_{D_T^2}\left[(v_{\Delta x}-v)D^+_{x_2}(u_{\Delta x}-u)D^+_{x_2}\Omega+(w_{\Delta x}-w)D^-_{y_2}(u_{\Delta x}-u)D^-_{y_2}\Omega\right]\,d\underline{z}}_{E}$$

$$+\underbrace{\frac{\kappa\Delta x}{2}\int_{D_T^2}(v_{\Delta x}+v_{\Delta x}(\cdot,x_2-\Delta x,\cdot)-2v)\left(D^+_{y_2}(w_{\Delta x}-w)+D^-_{x_2}(v_{\Delta x}-v)\right)D^-_{x_2}\Omega\,d\underline{z}}_{F}$$

$$+\underbrace{\frac{\kappa\Delta x}{2}\int_{D_T^2}(w_{\Delta x}+w_{\Delta x}(\cdot,y_2+\Delta x,\cdot)-2w)\left(D^+_{y_2}(w_{\Delta x}-w)+D^-_{x_2}(v_{\Delta x}-v)\right)D^+_{y_2}\Omega\,d\underline{z}}_{G}$$

$$+\underbrace{\frac{\kappa\Delta x}{2}\int_{D_T^2}D^+_{x_2}(u_{\Delta x}-u)^2D^+_{x_2}\Omega+D^+_{y_2}(u_{\Delta x}-u)^2D^+_{y_2}\Omega\,d\underline{z}}_{H}$$

$$+\underbrace{\frac{\kappa\Delta x^2}{4}\int_{D_T^2}\Big\{D^-_{x_2}\left(|D^+_{x_2}v_{\Delta x}|^2\right)+D^-_{y_2}\left(|D^-_{x_2}v_{\Delta x}|^2\right)-D^+_{x_2}\left(|D^+_{y_2}w_{\Delta x}|^2\right)-D^+_{y_2}\left(|D^-_{y_2}w_{\Delta x}|^2\right)\Big\}\Omega\,d\underline{z},}_{J}$$

where we have denoted $d\underline{z} := dt\,ds\,dx_1\,dx_2\,dy_1\,dy_2$. Estimating the terms A and D is done similarly to the one-dimensional wave equation, cf. Lemma 2.1. Thus,

$$A=\underbrace{\int_{D_T^2}\eta(u-u_{\Delta x},v-v_{\Delta x},w-w_{\Delta x},c)\psi_t^\mu\omega_\epsilon\omega_\epsilon\omega_{\epsilon_0}\,d\underline{z}}_{A_1}+\underbrace{\int_{D_T^2}\eta(u-u_{\Delta x},v-v_{\Delta x},w-w_{\Delta x},c)\psi^\mu\,\omega_\epsilon\omega_\epsilon\left(\partial_t\omega_{\epsilon_0}+D_s^-\omega_{\epsilon_0}\right)\,d\underline{z}}_{A_2} \tag{3.28}$$

Define

$$\lambda(t):=\frac{1}{2}\int_0^T\int_{D^2}\Big[|u_{\Delta x}-u|^2+\frac{1}{c}|v_{\Delta x}-v|^2+\frac{1}{c}|w_{\Delta x}-w|^2\Big]\omega_\epsilon\omega_\epsilon\omega_{\epsilon_0}\,dx_1dx_2dy_1dy_2ds,$$

we have

$$A_1=\int_0^T\lambda(t)\omega_\mu(t-\nu)\,dt-\int_0^T\lambda(t)\omega_\mu(t-\tau)\,dt, \tag{3.29}$$

and

$$|A_2|\leq C\Delta x\epsilon_0^{2\gamma-2}+\frac{C\Delta x}{\epsilon_0^{2-\gamma}}\int_0^T\sqrt{\lambda(t)}\psi^\mu\,dt, \tag{3.30}$$

using the estimates from Lemma 3.1 (c.f. the derivation of (2.35)). Similarly,

$$|D|\leq\frac{C\epsilon^\alpha}{\epsilon_0^{1-2\gamma}}+\frac{C\epsilon^\alpha}{\epsilon_0^{1-\gamma}}\int_0^T\sqrt{\lambda(t)}\,\psi^\mu\,dt, \tag{3.31}$$

(c.f. the derivation of (2.42)). In order to estimate the term B, we recall (**??**), and a similar identity for the forward difference,

$$
\begin{aligned}
D^-_{y_2}\Omega + \Omega_{y_1} &= \frac{1}{\Delta x}\int_0^{\Delta x} (\xi - \Delta x)\Omega_{y_2 y_2}(\cdot, y_2 - \xi)\, d\xi, \\
D^+_{x_2}\Omega + \Omega_{x_1} &= \frac{1}{\Delta x}\int_0^{\Delta x} (\Delta x - \xi)\Omega_{x_2 x_2}(\cdot, x_2 + \xi, \cdot)\, d\xi,
\end{aligned}
$$

hence,

$$
\begin{aligned}
B = \frac{1}{\Delta x}\int_0^{\Delta x}\int_{D_T^2} \big\{ & (u_{\Delta x} - u)(v_{\Delta x} - v)(\Delta x - \xi)\Omega_{x_1 x_1}(\cdot, x_2 + \xi, \cdot) \\
& + (u_{\Delta x} - u)(w_{\Delta x} - w)(\xi - \Delta x)\Omega_{y_1 y_1}(\cdot, y_2 - \xi)\big\}\, d\underline{z}\, d\xi := B_1 + B_2
\end{aligned}
$$

Estimating the terms B_1 and B_2 follows now along similar lines as estimating the term B in the one-dimensional case:

$$
\begin{aligned}
|B_1| &= \frac{1}{\Delta x}\bigg|\int_0^{\Delta x}\int_{D_T^2} \big\{ (u_{\Delta x} - u)(v_{\Delta x} - v) - (u_{\Delta x} - u(t, x_2, y_1))(v_{\Delta x} - v(t, x_2, y_1))\big\} \\
&\qquad\qquad \times (\Delta x - \xi)\Omega_{x_1 x_1}(\cdot, x_2 + \xi, \cdot)\, d\underline{z}\, d\xi\bigg| \\
&= \frac{1}{\Delta x}\bigg|\int_0^{\Delta x}\int_{D_T^2} \big\{ (u(t, x_2, y_1) - u)(v_{\Delta x} - v) + (u_{\Delta x} - u(t, x_2, y_1))(v - v(t, x_2, y_1))\big\} \\
&\qquad\qquad \times (\Delta x - \xi)\Omega_{x_1 x_1}(\cdot, x_2 + \xi, \cdot)\, d\underline{z}\, d\xi\bigg| \\
&\le \frac{1}{\Delta x}\int_0^{\Delta x}\int_{D_T^2} |u(t, x_2, y_1) - u||v_{\Delta x} - v|\, |\Delta x - \xi||\Omega_{x_1 x_1}|\, d\underline{z}\, d\xi \\
&\quad + \frac{1}{\Delta x}\int_0^{\Delta x}\int_{D_T^2} |u_{\Delta x} - u(t, x_2, y_1)||v - v(t, x_2, y_1)|\, |\Delta x - \xi||\Omega_{x_1 x_1}|\, d\underline{z}\, d\xi \\
&\le \frac{\Delta x}{\epsilon^{2-\gamma}} \underbrace{\int_0^T \sup_{\substack{x_1 \\ |x_1 - x_2| \le 3\epsilon}} \left(\int_{D_T}\int_{d_L^2}^{d_R^2} (v_{\Delta x} - v)^2 \omega_\epsilon \omega_{\epsilon_0}\, ds dy_1 dx_2 dy_2\right)^{1/2} \psi^\mu\, dt}_{b_1} \\
&\quad + \frac{\Delta x}{\epsilon^{2-\beta}}\int_0^T \left(\int_{D_T}\int_{d_L^2}^{d_R^2} (u_{\Delta x} - u(t, x_2, y_1))^2 \omega_\epsilon \omega_{\epsilon_0}\, ds dy_1 dx_2 dy_2\right)^{1/2} \psi^\mu\, dt
\end{aligned}
$$

Using then, c.f. (2.36), (2.37),

(3.32)

$$
\begin{aligned}
b_1 &\le \int_0^T \sup_{\substack{x_1 \\ |x_1 - x_2| \le \epsilon}} \bigg\{ \left(\int_{D_T}\int_{d_L^2}^{d_R^2} (v(t, x_1, y_1) - v(t, x_2, y_1))^2 \omega_\epsilon \omega_{\epsilon_0}\, ds dy_1 dx_2 dy_2\right)^{1/2} \\
&\qquad + \left(\int_{D_T}\int_{d_L^2}^{d_R^2} (v(t, x_2, y_1) - v_{\Delta x}(s, x_2, y_2))^2 \omega_\epsilon \omega_{\epsilon_0}\, ds dy_1 dx_2 dy_2\right)^{1/2}\bigg\} \psi^\mu\, dt \\
&\le CT\epsilon^\beta + \int_0^T \left(\int_{D_T}\int_{d_L^2}^{d_R^2} (v(t, x_2, y_1) - v_{\Delta x}(s, x_2, y_2))^2 \omega_\epsilon \omega_{\epsilon_0}\, ds dy_1 dx_2 dy_2\right)^{1/2} \psi^\mu\, dt \\
&\le CT\epsilon^\beta + \int_0^T \bigg(\int_{D_T}\int_{D} (v(t, x_2, y_1) - v_{\Delta x}(s, x_2, y_2))^2 \\
&\qquad\qquad \times \omega_\epsilon(y_1 - y_2)\omega_\epsilon(x_1 - x_2)\omega_{\epsilon_0}\, ds dx_1 dy_1 dx_2 dy_2\bigg)^{1/2} \psi^\mu\, dt
\end{aligned}
$$

$$
\begin{aligned}
&\le CT\epsilon^{\beta} + \int_0^T \Bigg\{ \bigg(\int_{D_T} \int_D (v(t,x_1,y_1) - v(t,x_2,y_2))^2 \omega_\epsilon \omega_\epsilon \omega_{\epsilon_0} \, dx_1 dy_1 dx_2 dy_2 ds \bigg)^{1/2} \\
&\qquad\qquad + \bigg(\int_{D_T} \int_D (v(t,x_1,y_1) - v_{\Delta x}(s,x_2,y_2))^2 \omega_\epsilon \omega_\epsilon \omega_{\epsilon_0} \, dx_1 dy_1 dx_2 dy_2 ds \bigg)^{1/2} \Bigg\} \psi^\mu \, dt \\
&\le CT\epsilon^{\beta} + \int_0^T \sqrt{\lambda(t)} \psi^\mu(t) \, dt,
\end{aligned}
$$

and

$$
\int_0^T \sup_{\substack{x_1 \\ |x_1 - x_2| \le 3\epsilon}} \bigg(\int_{D_T} \int_{d_L^2}^{d_R^2} (u_{\Delta x} - u)^2 \omega_\epsilon \omega_{\epsilon_0} \, ds dy_1 dx_2 dy_2 \bigg)^{1/2} \psi^\mu \, dt \le CT\epsilon^\gamma + \int_0^T \sqrt{\lambda(t)} \psi^\mu(t) \, dt,
$$

and estimating B_2 in a similar way, we find

$$
|B| \le C\Delta x \, \epsilon^{\gamma+\beta-2} + \frac{C\Delta x}{\epsilon^{2-\beta}} \int_0^T \sqrt{\lambda(t)} \, \psi^\mu \, dt, \tag{3.33}
$$

where $\beta = \min\{\alpha, \gamma\}$. It remains to estimate the terms E, F, G, H, I and J.

Estimate for E. We split E into two term, each of which can be estimated in the same way,

$$
\begin{aligned}
E = &\underbrace{\Delta x \int_{D_T^2} (v_{\Delta x} - v) D_{x_2}^+ (u_{\Delta x} - u) D_{x_2}^+ \Omega \, d\underline{z}}_{E_1} \\
&+ \underbrace{\Delta x \int_{D_T^2} (w_{\Delta x} - w) D_{y_2}^- (u_{\Delta x} - u) D_{y_2}^- \Omega \, d\underline{z}}_{E_2}.
\end{aligned}
$$

Using the Hölder inequality,

$$
\begin{aligned}
|E_1| &\le \int_{D_T^2} \bigg| (v_{\Delta x} - v) D_{x_2}^+ (u_{\Delta x} - u) \int_0^{\Delta x} \omega_\epsilon'(x_1 - x_2 - \xi) \omega_\epsilon(y_1 - y_2) \omega_{\epsilon_0} \psi^\mu \bigg| \, d\underline{z} d\xi \\
&\le \int_0^{\Delta x} \int_0^T \int_{d_L^1}^{d_R^1} \bigg(\int_{D_T} \int_{d_L^2}^{d_R^2} (v_{\Delta x} - v)^2 \omega_\epsilon \omega_{\epsilon_0} \, ds dy_1 dx_2 dy_2 \bigg)^{1/2} \\
&\qquad\qquad \times \bigg(\int_{D_T} \int_{d_L^2}^{d_R^2} \left(D_{x_2}^+ u_{\Delta x} \right)^2 \omega_\epsilon \omega_{\epsilon_0} \, ds dy_1 dx_2 dy_2 \bigg)^{1/2} |\omega_\epsilon'(x_1 - x_2 - \xi)| \, dx_1 \psi^\mu \, dt d\xi \\
&\le \int_0^{\Delta x} \int_0^T \sup_{\substack{x_1 \\ |x_1 - x_2| \le 3\epsilon}} \Bigg\{ \bigg(\int_{D_T} \int_{d_L^2}^{d_R^2} (v_{\Delta x} - v)^2 \omega_\epsilon \omega_{\epsilon_0} \, ds dy_1 dx_2 dy_2 \bigg)^{1/2} \Bigg\} \\
&\qquad \times \bigg(\int_{D_T} \left(D_{x_2}^+ u_{\Delta x} \right)^2 \, dx_2 dy_2 \bigg)^{1/2} \int_{d_L^1}^{d_R^1} |\omega_\epsilon'(x_1 - x_2 - \xi)| \, dx_1 \psi^\mu \, dt d\xi \\
&\le \frac{C\Delta x^\gamma}{\epsilon} \underbrace{\int_0^T \sup_{\substack{x_1 \\ |x_1 - x_2| \le 3\epsilon}} \Bigg\{ \bigg(\int_{D_T} \int_{d_L^2}^{d_R^2} (v_{\Delta x} - v)^2 \omega_\epsilon \omega_{\epsilon_0} \, ds dy_1 dx_2 dy_2 \bigg)^{1/2} \Bigg\} \psi^\mu \, dt}_{e_1}.
\end{aligned}
$$

The term e_1 is the same as the term b_1, hence it can be bounded by (3.32). The term E_2 is bounded by the same argument with the roles of v and w and x and y interchanged. Thus we arrive at the bound

$$
|E| \le \frac{C\Delta x^\gamma}{\epsilon^{1-\beta}} + \frac{C\Delta x^\gamma}{\epsilon} \int_0^T \sqrt{\lambda(t)} \psi^\mu(t) \, dt. \tag{3.34}
$$

Estimates for F, G and H. The terms F and G can be bounded using the same argument as for E, with $D^+_{y_2}(w_{\Delta x}-w)+D^-_{x_2}(v_{\Delta x}-v)$ taking the place of either $D^+_{x_2}(u_{\Delta x}-u)$ or $D^+_{y_2}(u_{\Delta x}-u)$ and using the triangle inequality to split them into two terms each. Hence they also satisfy (3.34). To estimate H, we observe that

$$\begin{aligned} D^+_{x_2}(u_{\Delta x}-u)^2 &= (u_{\Delta x}(s,x_2,y_2)-2u(t,x_1,y_1)+u_{\Delta x}(t,x_2+\Delta x,y_2))\,D^+_{x_2}u_{\Delta x} \\ &= (u_{\Delta x}(s,x_2,y_2)-u(t,x_1,y_1))\,D^+_{x_2}u_{\Delta x} \\ &\quad + (u_{\Delta x}(s,x_2+\Delta x,y_2)-u(t,x_1,y_1))\,D^+_{x_2}u_{\Delta x}. \end{aligned}$$

Splitting H according to this gives four terms, all of which can be estimated using the above arguments, hence also H satisfies (3.34).

Estimate for J: To estimate the term J, we observe that

$$\begin{aligned} &\int_{D_T^2} D^-_{x_2}\left(|D^+_{x_2}v_{\Delta x}|^2\right)\Omega\,d\underline{z} \\ &= \int_0^T\int_{D_T} D^-_{x_2}\left(|D^+_{x_2}v_{\Delta x}|^2\right)\int_D \omega_\epsilon(x_1-x_2)\omega_\epsilon(y_1-y_2)\,dx_1\,dy_1\,\psi^\mu\,dx_2\,dy_2\,dt\,ds \\ &= \int_0^T\int_0^T\sum_{ij} D^-_{x_2}\left(|D^+_{x_2}v_{\Delta x}(s,x_i,y_j)|^2\right)\psi^\mu\,dt\,ds \\ &= 0, \end{aligned}$$

since $\left(|D^+_{x_2}v_{\Delta x}|^2\right)$ is independent of x_1 and y_1. Using a similar argument for the terms containing $D^-_{y_2}\left(|D^-_{x_2}v_{\Delta x}|^2\right)$, $D^+_{x_2}\left(|D^+_{y_2}w_{\Delta x}|^2\right)$ and $D^+_{y_2}\left(|D^-_{y_2}w_{\Delta x}|^2\right)$, we find

$$J = 0.$$

Summing up

$$|E+F+G+H+J| \le \frac{C\Delta x^\gamma}{\epsilon^{1-\beta}} + \frac{C\Delta x^\gamma}{\epsilon}\int_0^T \sqrt{\lambda(t)}\psi^\mu(t)\,dt$$

Collecting the above equation and (3.29), (3.30), (3.31) and (3.33),

$$\begin{aligned} \int_0^T \lambda(t)\omega_\mu(t-\nu)\,dt &\le \int_0^T \lambda(t)\omega_\mu(t-\tau)\,dt \\ &\quad + C\underbrace{\left(\frac{\epsilon^\alpha}{\epsilon_0^{1-2\gamma}} + \frac{\Delta x}{\epsilon_0^{2-2\gamma}} + \frac{\Delta x^\gamma}{\epsilon^{1-\beta}} + \frac{\Delta x}{\epsilon^{2-\gamma-\beta}}\right)}_{M_1} \\ &\quad + C\underbrace{\left(\frac{\Delta x}{\epsilon_0^{2-\gamma}} + \frac{\epsilon^\alpha}{\epsilon_0^{1-\gamma}} + \frac{\Delta x^\gamma}{\epsilon} + \frac{\Delta x}{\epsilon^{2-\beta}}\right)}_{M_2}\int_0^T \sqrt{\lambda(t)}\psi^\mu(t)\,dt, \end{aligned} \tag{3.35}$$

where we have used that $0<\beta<1$, so that $\Delta x < \Delta x^\beta$, and that $\beta \le \gamma$, so that $\epsilon^{1-\gamma} \ge \epsilon^{1-\beta}$. Here, C is a constant depending on the moduli of continuity of the initial data, the $C^{0,\alpha}$-norm of c and on T. We can send μ to zero in (3.35)

$$\lambda(\nu) \le \lambda(\tau) + M_1 + M_2\int_\tau^\nu \sqrt{\lambda(t)}\,dt.$$

Again, applying the Gronwall inequality, [3, Chapter 1,Theorem 4], and sending $\mu \to 0$, we obtain the estimate

$$\lambda(\nu) \le \left(\sqrt{\lambda(\tau)+M_1} + (\nu-\tau)M_2\right)^2. \tag{3.36}$$

Repeating the arguments by which we obtained (2.47), (2.48) and (2.49), in two space dimensions, we arrive at

$$\|(u-u_{\Delta x})(\tau,\cdot)\|_{L^2(D)} + \|(v-v_{\Delta x})(\tau,\cdot)/c\|_{L^2(D)} + \|(w-w_{\Delta x})(\tau,\cdot)/c\|_{L^2(D)} \tag{3.37}$$

$$\leq C\Bigg(\|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/c\|_{L^2(D)} + \|(w_0 - w_{\Delta x}(0,\cdot))/c\|_{L^2(D)}$$
$$+ \epsilon_0^\gamma + \epsilon^\beta + \epsilon^{\alpha/2}\epsilon_0^{\gamma-1/2} + \frac{\Delta x^{1/2}}{\epsilon_0^{1-\gamma}} + \frac{\Delta x^{\gamma/2}}{\epsilon^{(1-\beta)/2}} + \frac{\Delta x^{1/2}}{\epsilon^{1-(\gamma+\beta)/2}} + \frac{\Delta x}{\epsilon_0^{2-\gamma}} + \frac{\epsilon^\alpha}{\epsilon_0^{1-\gamma}} + \frac{\Delta x^\gamma}{\epsilon} + \frac{\Delta x}{\epsilon^{2-\beta}}\Bigg).$$

Minimizing this expression over ϵ, ϵ_0, we find $\epsilon_0 = \epsilon^\alpha$ and $\epsilon = \Delta x^{\gamma/(\alpha\gamma+1)}$ if $\alpha \leq (2\gamma-1)/(\gamma(2-\gamma))$ and $\epsilon = \Delta x^{1/(\alpha\gamma+2-\alpha)}$ if $\alpha \geq (2\gamma-1)/(\gamma(2-\gamma))$. □

Corollary 3.1. *Under the assumptions of Lemma 3.2, the approximation $(u_{\Delta x}, \mathbf{r}_{\Delta x})$ converges to the solution $(u, \mathbf{r})$ ($\mathbf{r} = (r_1, r_2)$) of (3.1) at the rate*

$$\|u(t,\cdot) - u_{\Delta x}(t,\cdot)\|_{L^2(D)} + \|\mathbf{r}(t,\cdot) - \mathbf{r}_{\Delta x}(t,\cdot)\|_{L^2(D)} \leq C\left(\Delta x^s + \|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/c\|_{L^2(D)}\right), \tag{3.38}$$

for $0 < t < T$ where C is a constant depending on $\underline{c}$, $\overline{c}$, $\|c\|_{C^{0,\alpha}}$, T and on the L^2-norm of v_0, but not on Δx and s is given in (3.25).

Remark 3.1. *Defining an approximation for the primary variable p of the second order wave equation (1.1) by*

$$D_t^+ p_{ij}^n = u_{ij}^n, \quad i,j \in \mathbb{Z}, n = 1,\dots,N_T, \tag{3.39a}$$

$$p_{ij}^0 = \frac{1}{\Delta x^2}\int_{\mathcal{C}_{ij}} p_0(x,y)\,dx\,dy, \quad i,j \in \mathbb{Z}, \tag{3.39b}$$

one could show in the same way as it was done in Subsection 2.2.1 that the approximation $p_{\Delta x}$ defined as an interpolation of p_{ij}^n on the grid converges in $H^1([0,T]\times D) \cap C([0,T]; H^1(D))$ to the solution p of the wave equation (1.1) at the rate at least $\min\{\alpha, \gamma, s\}$, where s is given in (3.25).

3.1. **Numerical experiments.** Again, we compare the theoretically established rates to some practical examples. As the computational domain, we use the unit square $D = [0,1]^2$ with periodic boundary conditions. As a coefficient c, we again choose a sample of a log-normally distributed random field. As mentioned before, this is uniformly positive, bounded from above and Hölder continuous with exponent $1/2$. For a plot of the coefficient, see Figure 1.

We compute approximations at time $T = 0.5$, for $\kappa = 0.1$ to two sets of initial data

$$p_{0,1}(x) = \sin(2\pi x)\cos(2\pi y), \quad u_{0,1}(x) = \sin(2\pi x)\cos(2\pi x), \tag{3.40}$$

and

$$p_{0,2}(x) = \begin{cases} 2(x+y-0.5), & \text{if } x,y < 0.5,\ x+y \geq 0.5, \\ 2(1.5-x-y), & \text{if } x,y \geq 0.5,\ x+y \leq 1.5, \\ 2(y-x+0.5), & \text{if } x \geq 0.5,\ y < 0.5,\ y-x > 0.5, \\ 2(x-y+0.5), & \text{if } x < 0.5,\ y \geq 0.5,\ x-y > 0.5, \end{cases} \tag{3.41}$$

$$u_{0,2}(x) = \begin{cases} 1.5, & \text{if } x-y > 0.5,\ x+y > 0.5, \\ 0.5, & \text{otherwise}, \end{cases}$$

We notice that according to Lemma 3.1, the moduli of continuity of the variables u, v, w, r_1 and r_2 are for the initial data (3.40) and (3.41) at least $\gamma = 0.5$. In order to test the convergence, we have computed reference approximations on a grid with $N_{x,y} = 2^{11}$ gridpoints in each spatial direction. We have plotted the reference solutions to initial data (3.40) in Figure 4. We have approximated the numerical rate by (2.59) with $\mathcal{E}^2_{\Delta x_k}$ given as a two-dimensional version of (2.60).

We have used $\Delta x_0 = 1/8$ and $N_{\text{exp}} = 6$. For the initial data (3.40), the observed rate was ≈ 0.3 for the variables u, v, w, r_1 and r_2 and ≈ 0.45 for the variable p. For the initial data (3.41), we have observed a rate of ≈ 0.2 for the variables u, v and w, a rate of ≈ 0.17 for the variables r_1 and r_2 and a rate of ≈ 0.4 for p. Again, this is better than what (3.24), (3.25) predicts. Therefore we attempt, as before, to minimize expression (3.37) only for the given parameters $\alpha = \gamma = 1/2$ and

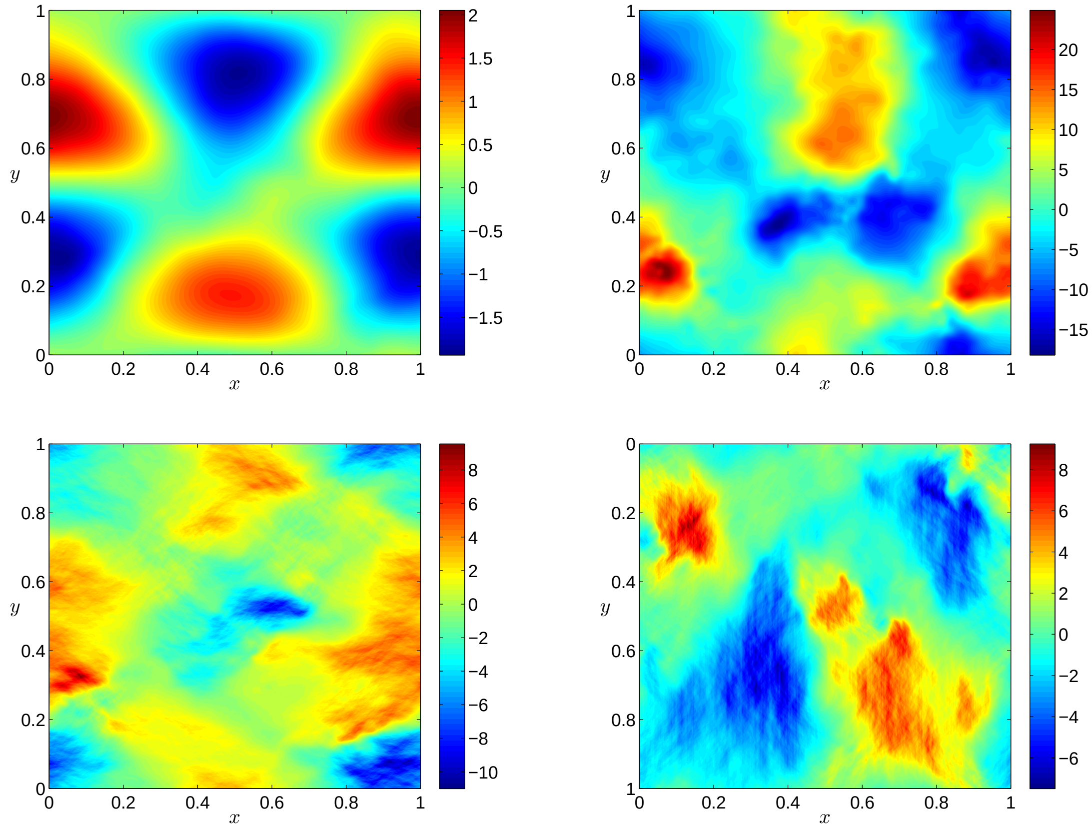

FIGURE 4. Top left: Approximation of the variable p in (3.2) with initial data (3.40) at time $T = 0.5$ on a mesh with 2^{11} grid cells in each direction. Top right: Approximation of the variable u. Bottom left: Approximation of the variable r_1. Bottom right: Approximation of the variable r_2.

Δx_k, $k = 0, \dots, 5$. We obtain an approximate rate of 0.19 which is quite close to what we observe numerically.

Remark 3.2. *By using similar techniques, we can obtain a slightly improved rate for a scheme using only central differences;*

$$D_t^+ u_{ij}^n = D_x^c v_{ij}^n + D_y^c w_{ij}^n + \Delta x \kappa \left({D_x^c}^2 + {D_y^c}^2 \right) u_{ij}^n \tag{3.42a}$$

$$\frac{1}{c_{ij}} D_t^+ v_{ij}^n = D_x^c u_{ij}^n + \Delta x \kappa {D_x^c}^2 v_{ij}^n + \Delta x \kappa D_x^c D_y^c w_{ij}^n \tag{3.42b}$$

$$\frac{1}{c_{ij}} D_t^+ w_{ij}^n = D_y^c u_{ij}^n + \Delta x \kappa D_x^c D_y^c v_{ij}^n + \Delta x \kappa {D_y^c}^2 w_{ij}^n. \tag{3.42c}$$

Our calculations are already lengthy we only report the end result here. The scheme (3.42) converges to the exact solution at the rate

$$\begin{cases} \Delta x^{\gamma/2}, & \alpha \geq \gamma,\ \alpha \geq 2/(2+\gamma), \\ \Delta x^{\alpha\gamma(1+\gamma)/(2+\alpha\gamma)}, & \alpha \geq \gamma,\ \alpha \leq 2/(2+\gamma), \\ \Delta x^{\alpha\gamma(1+\gamma)/(2+\alpha\gamma)}, & \alpha \leq \gamma,\ \alpha \geq (3\gamma-1)/(1+2\gamma-\gamma^2), \\ \Delta x^{2\alpha\gamma/(\alpha\gamma+3-\alpha)}, & \alpha \leq \gamma,\ \alpha \leq (3\gamma-1)/(1+2\gamma-\gamma^2), \end{cases} \tag{3.43}$$

provided a suitable CFL-condition, $\Delta t \leq \mathcal{O}(\Delta x)$, holds.

4. Conclusion

Acoustic waves that propagate in a heterogenous medium, for instance an oil and gas reservoir, are modeled using the linear wave equation (1.1) with a variable material coefficient c. Standard finite difference and finite element approximations are shown to converge to the solution as the mesh is refined. A rate of convergence for these approximations in obtained based on the assumption that the underlying solution is smooth enough. This requires enough smoothness on the material coefficient (wave speed).

However in many practical situations of interest such as seismic wave imaging and hydrocarbon exploration, the material coefficient is not smooth, not even continuously differentiable. On the other hand, the material coefficient (rock permeability) is usually modeled by a log-normal random field. Pathwise realizations of such fields are at most Hölder continuous. Thus, the design of numerical schemes that can approximate wave propagation in Hölder continuous media is a necessary first step in the efficient solution of the underlying uncertain PDE with a log-normal material coefficient [17]. Unfortunately, rigorous numerical analysis results for discretizations of the wave equation with such rough coefficients are not available currently.

The current paper is the first attempt to design robust numerical approximations for wave equations with rough i.e, Hölder continuous coefficients. We propose an *upwind* finite difference approximation for the wave equation with a Hölder continuous coefficient and show that these approximations converge as the mesh is refined. Furthermore, the key point of our paper is the rigorous determination of the convergence rates of these approximations. The obtained rates explicitly depend on the Hölder exponent of the material coefficient as well as the modulus of continuity in L^2 of the initial data. The rates of convergence are obtained by *novel* adaptation of the Kruzkhov doubling of variables technique from scalar conservation laws to our L^2 linear system setting. Numerical experiments demonstrating the near sharpness of the obtained rates are also presented.

We conclude with a brief discussion on possible limitations and future extensions of our methods:

- We consider finite difference discretizations in the current paper. The "formal" order of accuracy of our three-point finite difference schemes is one. One can argue that analogous to linear hyperbolic systems with smooth coefficients, one can obtain higher rates of convergence by designing schemes with a larger stencil (a higher formal order of accuracy). We find that prospect unlikely to hold in practice on account of the lack of smoothness of the coefficient. Furthermore, the irregularities of the coefficient are not localized. Hence, one cannot expect any localization of singularities of the solution and its derivatives. This is in marked contrast to nonlinear systems of conservation laws where discontinuities such as shocks and contact discontinuities separate smooth parts of the flow. Thus, high-resolution finite difference schemes perform better than low order schemes for conservation laws. Such a situation does not hold for wave propagation in a rough medium. We expect that the low-order schemes presented here are not only simple but also optimal in this case.
- We present the analysis in both one and two space dimensions. The extension to three space dimensions is straightforward. However, our methods are restricted to Cartesian grids. In principle, one can expect to adapt these methods to structured grids. However, an extension to unstructured grids in several space dimensions presents a challenge. One can expect to design a finite volume type method (our algorithms can also be thought of finite volume methods as we approximate cell averages) on unstructured grids and prove rates of convergence. It can be considered as a forthcoming project.
- We restrict ourselves to acoustic wave propagation in rough media in this paper. However, elastic wave propagation also involve media with material properties that lead to rough, Hölder continuous coefficients. The extension of these methods to such problems will be considered in a forthcoming paper. Another possible direction of research would be prove rate of convergence for numerical methods that approximate electromagnetic wave propagation in heterogeneous media. Possible extensions to nonlinear wave propagation can also be considered.

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