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Isotropic Gaussian random fields on the sphere

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(joint work with Christoph Schwab)

Sample regularity and fast simulation of isotropic Gaussian random fields on the sphere are for example of interest for the numerical analysis of stochastic partial differential equations and for the simulation of ice crystals or Saharan dust particles as lognormal random fields. In what follows we recall the results from [2], which include the approximation of isotropic Gaussian random fields with convergence rates as well as the regularity of the samples in relation to the smoothness of the covariance expressed in terms of the decay of the angular power spectrum. As example we construct isotropic Q-Wiener processes out of isotropic Gaussian random fields and discretize the stochastic heat equation with spectral methods.

Before we state the results, we start with a short review of the basics. Therefore, let $(\Omega, \mathcal{A}, (\mathcal{F}_t), P)$ be a filtered probability space and denote by $\mathbb{S}^2 \subset \mathbb{R}^3$ the unit sphere. A $\mathcal{A} \otimes \mathcal{B}(\mathbb{S}^2)$-measurable mapping $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ is called an *isotropic Gaussian random field* if, for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{S}^2$, $a_1, \ldots, a_k \in \mathbb{R}$, the real-valued random variable $\sum_{i=1}^k a_i T(x_i)$ is Gaussian and the distribution of $(T(x_1), \ldots, T(x_k))$ is invariant under rotations. By [3], the isotropic Gaussian random field T admits a Karhunen–Loève expansion $T = \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}$, where $(Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \ldots, \ell)$ denotes the sequence of spherical harmonic functions and $(a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \ldots, \ell)$ is a sequence of normally distributed random variables, whose properties are characterized by the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. For $\ell \in \mathbb{N}$, $m = 1, \ldots, \ell$, and $\vartheta \in [0, \pi]$, let

$$L_{\ell m}(\vartheta) := \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(\cos\vartheta)$$

be a weighted version of the associate Legendre polynomials $(P_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$. Then the random field generated by

$$\sum_{\ell=0}^{\infty} \Big(\sqrt{A_\ell} X^1_{\ell 0} L_{\ell 0}(\vartheta) + \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(X^1_{\ell m} \cos(m\varphi) + X^2_{\ell m} \sin(m\varphi)) \Big) + \mathbb{E}(T)$$

is equal in law to T, where $((X^1_{\ell m}, X^2_{\ell m}), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ is a sequence of independent standard normally distributed random variables with $X^2_{\ell 0} = 0$ for all $\ell \in \mathbb{N}_0$. A truncation of the series expansion leads to the following convergence results which rely on the decay of the angular power spectrum.

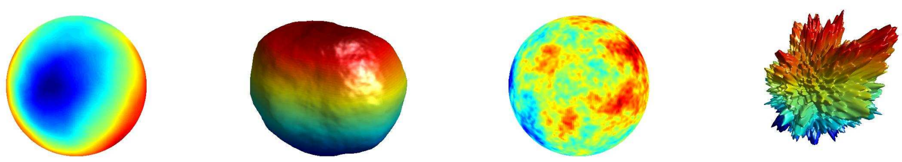
Theorem 1. *Assume that $A_\ell \leq C \cdot \ell^{-\alpha}$ for some $\alpha > 2$ and $C > 0$. Then for all $0 < p < +\infty$ there exists $\hat{C}_p > 0$ such that*

$$\|T - T^\kappa\|_{L^p(\Omega; L^2(\mathbb{S}^2))} \leq \hat{C}_p \cdot \kappa^{-(\alpha-2)/2},$$

where the truncated series expansion T^κ is given by

$$\sum_{\ell=0}^{\kappa} \Big(\sqrt{A_\ell} X^1_{\ell 0} L_{\ell 0}(\vartheta) + \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(X^1_{\ell m} \cos(m\varphi) + X^2_{\ell m} \sin(m\varphi)) \Big) + \mathbb{E}(T).$$

Furthermore, for all $\beta < (\alpha-2)/2$ it holds asymptotically $\|T - T^\kappa\|_{L^2(\mathbb{S}^2)} \leq \kappa^{-\beta}$, P-a. s..



(a) $\alpha = 5$ (b) $\alpha = 5$ (c) $\alpha = 3$ (d) $\alpha = 3$

FIGURE 1. Samples of isotropic Gaussian and the corresponding lognormal random fields as radius of the deformed sphere with $A_\ell = (\ell+1)^{-\alpha}$.

The decay of the angular power spectrum is linked to the regularity of the covariance kernel in the following proposition and can be extended to non-integers with fractional weighted Sobolev spaces.

Proposition 2. *For every $n \in \mathbb{N}_0$, it holds that the sequence $(\ell^{n+1/2} A_\ell, \ell \geq n)$ is in $\ell^2(\mathbb{N}_0)$ if and only if the covariance kernel $(1-\mu^2)^{n/2} \frac{\partial^n}{\partial \mu^n} \sum_{\ell=0}^\infty A_\ell \frac{2\ell+1}{4\pi} P_\ell(\mu)$ is in $L^2(-1,1)$, where $(P_\ell, \ell \in \mathbb{N}_0)$ denotes the sequence of Legendre polynomials.*

Furthermore, the decay of the angular power spectrum determines the sample regularity of the random field.

Theorem 3. *Assume that $\sum_{\ell=0}^\infty A_\ell \ell^{1+\beta} < +\infty$ for some $\beta > 0$. Then there exists a continuous modification of T which is Hölder continuous with exponent γ for all $\gamma < \min\{\beta/2, 1\}$. Furthermore, the modification is k-times continuously differentiable for all $k < \beta/2 - 1$. The corresponding* lognormal random field $\exp(T)$ *has the same regularity properties.*

The Hölder continuity in the previous theorem is proven using the following lemma and a version of the Kolmogorov–Chentsov theorem, which we state for completeness, while the differentiability is a direct consequence of Sobolev embeddings. The same regularity of the lognormal random field results from the properties of the exponential function. Samples of Gaussian and the corresponding lognormal random fields are shown in Figure 1.

Lemma 4. *Assume that $\sum_{\ell=0}^\infty A_\ell \ell^{1+\beta} < +\infty$ for some $\beta \in [0,2]$. Then the corresponding kernel function $k(r) = \sum_{\ell=0}^\infty A_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos r)$ satisfies that*

$$|k(0) - k(r)| \leq C_\beta r^\beta$$

for some $C_\beta > 0$, which implies that for all $0 < p < +\infty$ there exists $C_{\beta,p} > 0$ such that

$$\mathbb{E}(|T(x) - T(y)|^{2p}) \leq C_{\beta,p}\, d(x,y)^{\beta p}.$$

The second step in the proof is the Kolmogorov–Chentsov theorem for random fields on $\mathbb{S}^2$ which is proven by applying a version of the theorem for domains on six charts and patching the resulting random fields together with a partition of unity. This is extended to general manifolds and from Hölder continuity to Hölder differentiability in [1].

Theorem 5 (Kolmogorov–Chentsov theorem)**.** *Let T be a random field on $\mathbb{S}^2$ that satisfies*

$$\mathbb{E}(|T(x) - T(y)|^p) \leq C d(x,y)^{2+\epsilon p}$$

for some $p > 0$, $C > 0$, and some $\epsilon \in (0,1]$. Then there exists a continuous modification of T that is locally Hölder continuous with exponent γ for all $\gamma \in (0,\epsilon)$.

Besides the already mentioned application to lognormal random fields, isotropic Gaussian random fields can also be used to define a Q-Wiener process W taking values in $L^2(\mathbb{S}^2)$ by the Karhunen–Loève expansion

$$\sum_{\ell=0}^{\infty} \sqrt{A_\ell}\beta^1_{\ell 0}(t)L_{\ell 0}(\vartheta) + \sqrt{2A_\ell}\sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(\beta^1_{\ell m}(t)\cos(m\varphi) + \beta^2_{\ell m}(t)\sin(m\varphi)),$$

where $((\beta^1_{\ell m}, \beta^2_{\ell m}), \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$ is a sequence of independent Brownian motions and $\beta^2_{\ell 0} = 0$ for $\ell \in \mathbb{N}_0$. Here, the covariance operator Q is characterized by $QY_{\ell m} = A_\ell Y_{\ell m}$. Let us observe that the Laplace–Beltrami operator $\Delta_{\mathbb{S}^2}$ on $\mathbb{S}^2$ satisfies that $\Delta_{\mathbb{S}^2} Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$. We want to simulate the stochastic heat equation on $\mathbb{S}^2$ driven by additive Q-Wiener noise on some finite time interval

$$dX(t) = \Delta_{\mathbb{S}^2} X(t)\,dt + dW(t)$$

with initial condition $X(0) = X_0 \in L^2(\Omega; L^2(\mathbb{S}^2))$, i. e., in mild form

$$X(t) = X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\,ds + \int_0^t dW(s) = X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\,ds + W(t).$$

This equation can be expanded with respect to the spherical harmonic functions and leads to the stochastic differential equations

$$(X(t), Y_{\ell m})_{L^2(\mathbb{S}^2)} = (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} - \ell(\ell+1)\int_0^t (X(s), Y_{\ell m})_{L^2(\mathbb{S}^2)}\,ds + a_{\ell m}(t)$$

with scaled Brownian motions $a_{\ell m}$, which can be solved with the variations of constants formula. We are able to simulate the solution with the observation that the stochastic convolutions $\int_0^t e^{-\ell(\ell+1)(t-s)}\,d\beta_{\ell m}(s)$ are by the Itô formula normally distributed with mean zero and variance $(2\ell(\ell+1))^{-1}(1 - e^{-2\ell(\ell+1)t})$. In what follows, we obtain convergence results under weaker assumptions on $(A_\ell, \ell \in \mathbb{N}_0)$ than in Theorem 1 due to the smoothing of the heat kernel.

Theorem 6. *Assume that $A_\ell \leq C \cdot \ell^{-\alpha}$ for some $\alpha > 0$ and $C > 0$. Then, for all $0 < p < +\infty$, $t < +\infty$, and $\kappa \in \mathbb{N}$,*

$$\|X(t) - X^\kappa(t)\|_{L^p(\Omega; L^2(\mathbb{S}^2))} \leq \hat{C}_p \cdot \kappa^{-\alpha/2}$$

with $\hat{C}_p > 0$ independent of the time discretization, where X^κ denotes the truncated Karhunen–Loève expansion of the solution of the stochastic heat equation. Furthermore, for all $\beta < \alpha/2$ it holds asymptotically $\|X(t) - X^\kappa(t)\|_{L^2(\mathbb{S}^2)} \leq \kappa^{-\beta}$, P-a. s..

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