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Abstract

We consider the problem of approximating manifold-valued functions with approximation spaces spanned by linear combinations of cardinal B-splines with control points constrained to lie on the manifold, followed by a closest-point projection onto the manifold. Under certain conditions we can prove that these spaces realize the optimal approximation rate. Applications for denoising of manifold-valued data and the computation of geometric variational problems are discussed.

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1 Introduction

This papers considers quasiinterpolation operators for the approximation of functions which map into a Riemannian manifold.

The approximation of such functions is of increasing importance in a variety of applications among which we only mention exemplarily the processing of nonlinear data [15, 14], model reduction [2], or the solution of geometric partial differential equations [17, 18].

In the last years several results have been achieved in this direction, see e.g., [20, 22, 8, 13, 11, 10]. One common feature of all these works is that they generalize known linear approximation methods to a manifold-valued setting and that these linear methods are based on *interpolation* e.g., function values are used as control points in the approximation procedure.

This poses severe restrictions on the type of linear methods that can be used as a basis for a generalization. For instance classical approximation methods such as spline quasiinterpolation [3], Clément-type approximation [5], or isogeometric methods [6] cannot be generalized to a manifold-valued setting with the methods developed in the articles mentioned above. Nevertheless, having optimal error estimates for suitable generalizations of these methods would be highly desirable, for instance in solving elliptic geometric partial differential equations in variational form in the spirit of [12].

In the present article we achieve a first step towards this goal. More precisely we show how to appropriately extend the linear theory of periodic cardinal B-spline quasiinterpolation [3] to a manifold-valued setting, if the manifold is embedded into Euclidean spaces as the zero set of a smooth submersion. Various aspects of a similar problem from a subdivision-perspective are discussed in [22] where it is observed that a naive approach fails.

Let us now become a bit more concrete. Suppose we want to numerically approximate a function $\mathbf{f}:\Omega\to M$, where Ω is a domain in $\mathbb{R}^d$ and M is a smooth manifold, embedded into Euclidean space $\mathbb{R}^K$. The function $\mathbf{f}$ can be given explicitly or implicitly as a solution to an inverse problem [12]. If M were a linear space, say, $M=\mathbb{R}^K$ one would typically try to approximate $\mathbf{f}$ from a linear function space, say, $V:=\left\{\sum_i c_i\Phi_i(x):\, c_i\in\mathbb{R}^K\right\}$, where $\Phi:\Omega\to\mathbb{R}$ and study the resulting approximation error. For $M\neq\mathbb{R}^K$, the elements of V clearly do not map into M, so we modify the approximation space V to $V_M:=\{\mathbf{P}_M\left(\sum_i c_i\Phi_i(x)\right):\, c_i\in M\}$ ($\mathbf{P}_M$ denoting the closest-point projection onto M) which now consists of M-valued functions. These approximation spaces are particularly simple to handle numerically whenever $\mathbf{P}_M$ is simple to compute, and thus, in such situations they are, from a computational point

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of view, more attractive than geodesic finite elements [17, 18]. However, for the spaces V_M to be of any interest it is crucial to examine their approximation properties. Ideally they should match the well-studied properties of the corresponding linear spaces V. Indeed, it is not very difficult to verify this latter property, *whenever the functions Φ_i are exact on polynomials*[1]. This includes for example Lagrange finite element functions [4] or integer translates of Deslauriers-Dubuc scaling functions [7]. However, it excludes several important constructions such as those based on B-splines, NURBS or Bernstein-Bézier finite elements where polynomial exactness does not hold.

The present paper presents a first step towards establishing optimal approximation properties of spaces V_M, also if polynomial exactness does not hold, and thereby for the first time achieves a generalization of such methods to a manifold-valued setting. More precisely, we show (in a periodic setting) that for approximation spaces V generated by cardinal B-spline functions, the corresponding spaces V_M possess the same approximation properties as V.

The proof of this result turns out to be surprisingly subtle. The main technical novelty which makes our approach work is the introduction of the concept of *normal perturbation family*, see Section 3.2, which will serve as a crucial tool in proving our main results. Roughly speaking, a normal perturbation family is a perturbation $\tilde{\mathbf{f}}$ of $\mathbf{f}$, orthogonal to M, such that the control points $c_i(\tilde{\mathbf{f}})$ of an optimal approximation $\sum_i c_i(\tilde{\mathbf{f}})\Phi_i$ of $\tilde{\mathbf{f}}$ in V all lie in M, i.e. $c_i(\tilde{\mathbf{f}}) \in M$ (usually one is interested in a family of approximation spaces rather than a single one, hence the name normal perturbation family). This implies that the function $\mathbf{P}_M(\sum_i c_i(\tilde{\mathbf{f}})\Phi_i(x))$ lies in V_M and, assuming that $\tilde{\mathbf{f}}$ is as smooth as $\mathbf{f}$ it is not difficult to show (using Lipschitz-continuity of the composition operator with a smooth function) that the approximation error

$$\mathbf{f} - \mathbf{P}_M(\sum_i c_i(\tilde{\mathbf{f}})\Phi_i) = \mathbf{P}_M(\tilde{\mathbf{f}}) - \mathbf{P}_M(\sum_i c_i(\tilde{\mathbf{f}})\Phi_i)$$

is of the same order as the linear error $\tilde{\mathbf{f}} - \sum_i c_i(\tilde{\mathbf{f}})\Phi_i(x)$, which can be estimated by well-known linear results, assuming that $\tilde{\mathbf{f}}$ is as smooth as the original function $\mathbf{f}$. The key to make this idea work is to prove that such normal perturbation families exist. Our main result is that for spaces which are generated by translates of B-splines this holds true.

We believe that the idea outlined above and the notion of normal perturbation family will be of use in a much broader setting and eventually enable us to generalize a much larger class of linear approximation results to the manifold-valued case, and thus, using the results of [12], enable the use of (suitable generalizations of) isogeometric methods or p-methods based on Bernstein-Bézier polynomials [1] also for manifold-valued PDEs.

We have structured the article as follows. In Section 2 we introduce our notation and review some well-known facts related to embedded manifolds and linear quasiinterpolation. In Section 3 we first show that a naive generalization of linear quasiinterpolation operators cannot go beyond a certain (low) order of convergence. After this we introduce the notion of normal perturbation family and show that the existence of such families implies the existence of quasiinterpolation operators which have the desired approximation order. Section 4 contains the main technical part of this article. In it we establish the existence of normal perturbation families for spaces generated by B-splines in a periodic setting. Finally in section 5 we discuss some computational aspects and present some numerical examples and applications.

2 Basic Notions

In the present section we introduce the basic setting of this work, starting with some elementary facts on implicit manifolds in Section 2.1. After that, in Section 2.2 we review some aspects of the classical theory of linear quasiinterpolation with cardinal B-splines.

Before we proceed we briefly comment on the notation of this paper. As a general rule we shall use boldface $\mathbf{x}$ for elements in Euclidean space, capital boldface letters $\mathbf{M}$ for matrices in Euclidean space, Fraktur letters $\mathfrak{f}$ for sequences and capital Fraktur letters $\mathfrak{F}$ for operators on sequence spaces. We use the usual Notation $A = \mathcal{O}(B)$ or $A \lesssim B$ to indicate that the quantity A is bounded by a constant times B.

[1] The system (Φ_i) has polynomial exactness of order d if there exist nodes $\zeta_i \in \Omega$ such that for every polynomial p of degree $\leq d$ we have the exactness property $\sum_i p(\zeta_i)\Phi_i(x) = p(x)$.

2.1 Manifold Basics

We consider manifolds M which are defined implicitly, e.g.,

$$M = \left\{ \mathbf{x} \in \mathbb{R}^K : \mathbf{g}(\mathbf{x}) = \mathbf{0} \in \mathbb{R}^D \right\},$$

where

$$\mathbf{g}(\mathbf{x}) = \left(g^1(\mathbf{x}), \ldots, g^D(\mathbf{x}) \right)^T \in \mathbb{R}^D$$

with $g^i \in C^\infty(\mathbb{R}^K, \mathbb{R})$ for all $i \in \{1, 2, \ldots, D\}$ and

$$D\mathbf{g}(\mathbf{x}) := \left(\nabla g^1(\mathbf{x}), \ldots, \nabla g^D(\mathbf{x}) \right) \in \mathbb{R}^{K \times D} \tag{1}$$

is of constant rank D for all $\mathbf{x} \in M$. The normal space at a point $\mathbf{x} \in M$ is spanned by the vectors $(\nabla g^r(\mathbf{x}))_{r=1}^D$. In the course of our paper it will turn out beneficial to use a different basis of this normal space, as we now describe.

Due to (1) there exists a dual basis

$$\mathbf{N}(x) := (\mathbf{n}^1(\mathbf{x}), \ldots, \mathbf{n}^D(\mathbf{x})) \in \mathbb{R}^{K \times D}$$

of the normal space such that we have

$$D\mathbf{g}(\mathbf{x})^T \mathbf{N}(x) = \mathbf{I}_D \in \mathbb{R}^{D \times D}, \tag{2}$$

where $\mathbf{I}_D$ denotes the identity matrix in $\mathbb{R}^{D \times D}$. To see this, simply consider the matrix

$$\mathbf{A}(\mathbf{x}) := \left(\left\langle \nabla g^i(\mathbf{x}), \nabla g^j(\mathbf{x}) \right\rangle \right)_{i,j=1}^D.$$

By the assumption that $D\mathbf{g}$ has full rank, the matrix $\mathbf{A}(\mathbf{x})$ is invertible and therefore we can consider its inverse $\mathbf{B}(\mathbf{x}) := (b_{i,j}(\mathbf{x}))_{i,j=1}^D := \mathbf{A}^{-1}(\mathbf{x})$. Defining

$$\mathbf{n}^i(\mathbf{x}) := \sum_{j=1}^D b_{i,j}(\mathbf{x}) \nabla g^j(\mathbf{x}) \quad \text{for} \quad i = 1, \ldots, D$$

yields the dual basis.

Since $\mathbf{g} \in C^\infty(\mathbb{R}^K, \mathbb{R}^D)$, the mapping $x \in M \mapsto \mathbf{N}(x) \in \mathbb{R}^{N \times D}$ is smooth (i.e. C^∞).

We denote the closest-point projection

$$\mathbf{P}_M : \mathbb{R}^K \to M.$$

As $\mathbf{g} \in C^\infty(\mathbb{R}^K, \mathbb{R})$ the function $\mathbf{P}_M$ is smooth in a neighborhood of M.

2.2 Quasiinterpolation

After defining the necessary function spaces in Section 2.2.1, In the following Section 2.2.2 we give a condensed summary of the classical theory of quasiinterpolation of periodic functions on regular grids. All the material up to this point is completely classical.

2.2.1 Function Spaces

We begin by introducing the necessary function spaces we shall be working with. For much more information on function spaces we refer to the comprehensive books [16, 9].

Denote $\mathbb{T} := [0, 1)$ the one-dimensional torus. For $K \in \mathbb{N}$ and $m \in \mathbb{N}$ define the periodic Sobolev spaces

$$W^{m,p}(\mathbb{T}, \mathbb{R}^K) := \left\{ \mathbf{u} : \mathbb{T} \to \mathbb{R}^K : \|\mathbf{u}\|_{W^{m,p}(\mathbb{T},\mathbb{R}^K)} < \infty \right\},$$

where

$$\|\mathbf{u}\|_{W^{m,p}(\mathbb{T},\mathbb{R}^K)} := \|\mathbf{u}\|_{L^p(\mathbb{T},\mathbb{R}^K)} + \sum_{l=1}^m \left\| \frac{d^l}{dx^l} \mathbf{u} \right\|_{L^p(\mathbb{T},\mathbb{R}^K)},$$

and the differentiation operator is understood as acting on periodic functions. The space of continuous functions from the unit interval to $\mathbb{R}^K$ is denoted $C(\mathbb{T}, \mathbb{R}^K)$.

For $m > \frac{1}{p}$ we denote the spaces

$$W^{m,p}(\mathbb{T}, M) := \{\mathbf{u} \in W^{m,p}(\mathbb{T}, \mathbb{R}^K) : \ u(x) \in M \quad \text{for all } x \in \mathbb{T}\}$$

(observe that under our assumptions all these functions are continuous by the Sobolev embedding theorem [9], so this definition makes sense). We also write $C(\mathbb{T}, M)$ for all continuous M-valued functions.

We shall also require some discrete analogues of continuous function space norms which we are now going to describe. For notational convenience we sometimes for $n \in \mathbb{N}$ write

$$[n] := \{0, \dots, n-1\}.$$

For $K \in \mathbb{N}$ we denote the space $\ell\left([n], \mathbb{R}^K\right)$ of all functions $\mathfrak{f} : [n] \to \mathbb{R}^K$. For a sequence $\mathfrak{f} = (\mathbf{f}_i)_{i \in [n]} \in \ell\left([n], \mathbb{R}^K\right)$ we define the norms

$$\|\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K)} = \left(\sum_{i \in [n]} |\mathbf{f}_i|^p\right)^{1/p}, \quad \|\mathfrak{f}\|_{\ell^\infty([n],\mathbb{R}^K)} := \max_{i \in [n]} |\mathbf{f}_i|,$$

where we denote

$$|\mathbf{x}| := \sqrt{x_1^2 + \dots + x_K^2}, \quad \mathbf{x} = (x_1, \dots, x_K)^T \in \mathbb{R}^K$$

and $p \in [1, \infty]$.

We shall also require the operator $\nabla : \ell\left([n], \mathbb{R}^K\right) \to \ell\left([n], \mathbb{R}^K\right)$, defined by

$$\nabla \mathfrak{f}(i) = \begin{cases} \mathbf{f}_i - \mathbf{f}_{i-1} & 0 < i \leq n-1 \\ \mathbf{f}_0 - \mathbf{f}_{n-1} & i = 0 \end{cases}$$

and, for $p \in [1, \infty]$, $m > 0$, the (quasi) norms

$$\|\mathfrak{f}\|_{\ell^{m,p}([n],\mathbb{R}^K)} := \|n^m \nabla^{\lceil m \rceil} \mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K)}. \tag{3}$$

We will sometimes not include the domain or codomain in our notation for describing function spaces, i.e., we may sometimes write $W^{m,p}$ instead of $W^{m,p}(\mathbb{T}, \mathbb{R}^K)$ if the missing information can be easily inferred from the context.

2.2.2 Linear Quasiinterpolation

We now review the nuts and bolts of linear quasiinterpolation for periodic functions over a regular grid. More information on this topic can be found in [3]. It is well-known that approximation order is intimately related to the property of polynomial generation which we introduce in the following assumption.

Assumption 2.1. *For $d \in \mathbb{N}$ we assume the existence of a function $\Phi(x) : \mathbb{R} \to \mathbb{R}$ with compact support and a sequence $\Lambda = (\lambda_k)_{k=-S}^S$ such that we have the* polynomial generation property

$$\sum_{i \in \mathbb{Z}} \sum_{k=-S}^{S} \lambda_k u(i-k) \Phi(x-i) = u(x) \quad \textit{for all } u \in \Pi_d, \tag{4}$$

where Π_d denotes all polynomials of degree less than d.

Example 2.2. *We define cardinal B-splines recursively by*

$$B_0(x) = \begin{cases} 1 & \textit{if } -.5 \leq x < .5 \\ 0 & \textit{otherwise} \end{cases} \qquad \textit{and} \qquad B_k(x) = \int_{x-1/2}^{x+1/2} B_{k-1}(y) dy.$$

For a positive integer k we choose $\phi = B_k$. It can be shown that for odd k there is a unique sequence Λ of length k such that the polynomial generation property (4) is satisfied with $d = k+1$, e.g. $\Lambda = (-1/6, 4/3, -1/6)$ if $k = 3$ and $\Lambda = (13/240, -7/15, 73/40, -7/15, 13/240)$ if $k = 5$.

In this article we consider the approximation of periodic functions. Therefore, some trivial adaptions to the non-periodic theory need to be made, in particular we need to periodize the basis function Φ. To this end we write for $n \in \mathbb{N}$

$$\Phi_n^{\mathrm{PER}}(x) := \sum_{j\in\mathbb{N}} \Phi(n(x-j)), \quad x \in \mathbb{T},$$

the periodization of $\Phi(n\cdot)$.

The following definition introduces our notation for the *sampling operator* which extracts samples of a function $\mathbf{f}$.

Definition 2.3. *For $\mathbf{f} \in C(\mathbb{T}, \mathbb{R}^K)$ and $n \in \mathbb{N}$ define the* sampling

$$\mathcal{S}_n(\mathbf{f}) := \mathfrak{f}^n = \left(\mathbf{f}(\frac{i}{n})\right)_{i\in[n]} \in \ell([n], \mathbb{R}^K).$$

Finally we introduce the following notation for the well-known operation of circular convolution.

Definition 2.4. *For Λ as in Assumption* (2.1) *and n, $K \in \mathbb{N}$ we define the* circular convolution *as the linear operator*

$$circ_\Lambda : \begin{cases} \ell([n], \mathbb{R}^K) & \to & \ell([n], \mathbb{R}^K) \\ \mathfrak{f} & \mapsto & \left(\sum_{k=-S}^{S} \lambda_k \mathfrak{f}_{i-k \ mod \ n}\right)_{i\in[n]} \end{cases} \tag{5}$$

Now we have introduced all necessary technical tools to define the approximation spaces we are interested in.

Definition 2.5. *Define the approximation spaces*

$$V_{\mathbb{R}^K}^n := \left\{ \sum_{i\in[n]} c_i \Phi_n^{PER}(x - i/n) : (c_i)_{i\in[n]} \subset \mathbb{R}^K \right\}.$$

Define the linear Quasiinterpolation operator

$$Q_{\mathbb{R}^K}^n : \begin{cases} C(\mathbb{T}, \mathbb{R}^K) & \to & V_{\mathbb{R}^K}^n \\ \mathbf{f} & \mapsto & \sum_{i\in[n]} circ_\Lambda \mathfrak{f}_i^n \Phi_n^{PER}(x - i/n). \end{cases}$$

The following classical result relates the smoothness of $\mathbf{f}$ and the degree of polynomial generation of Φ to the asymptotic approximation rate of $Q_{\mathbb{R}^K}^n \mathbf{f}$ in n.

Theorem 2.6. *For $f \in W^{m,p}(\mathbb{T}, \mathbb{R}^K)$ and $m > \frac{1}{p}$ we have that*

$$\|\mathbf{f} - Q_{\mathbb{R}^K}^n \mathbf{f}\|_{W^{l,p}(\mathbb{T},\mathbb{R}^K)} \lesssim n^{l-m} \|\mathbf{f}\|_{W^{m,p}(\mathbb{T},\mathbb{R}^K)},$$

whenever $d \geq m - l$ and $\Phi \in W^{l,\infty}(\mathbb{T}, \mathbb{R}^K)$.

Proof. We only sketch the proof, which follows well-known arguments. We can assume that $f \in C^\infty(\mathbb{T}, \mathbb{R}^K)$. The theorem then follows by a standard density argument. It is enough to prove for all $r \leq l$ that

$$\left\| \frac{d^r}{dx^r} (\mathbf{f}(x) - Q_{\mathbb{R}^K}^n \mathbf{f}(x)) \right\|_{L^p} \lesssim n^{l-m} \|\mathbf{f}\|_{W^{m,p}}.$$

Since $\frac{d}{dx} Q_{\mathbb{R}^K}^n = Q_{\mathbb{R}^K}^n \frac{d}{dx}$ this is equivalent to

$$\left\| \frac{d^r}{dx^r} \mathbf{f}(x) - Q_{\mathbb{R}^K}^n \frac{d^r}{dx^r} \mathbf{f}(x) \right\|_{L^p} \lesssim n^{l-m} \|\mathbf{f}\|_{W^{m,p}}.$$

With the substitution $\tilde{\mathbf{f}}_r(x) := \frac{d^r}{dx^r} \mathbf{f}(x)$ it is enough to prove the theorem with $(\tilde{m}, \tilde{l}) = (m - l, 0)$. Hence we can restrict ourselves to the case $l = 0$.

Assume that $nx \in [j, j+1)$. Since the function Φ is compactly supported, there exists $R \in \mathbb{Z}$ such that

$$Q_{\mathbb{R}^K}^n \mathbf{f}(x) = \sum_{i=j-R}^{j+R} \sum_{k=-S}^{S} \lambda_k \mathbf{f}\left(\frac{i-k}{n}\right) \Phi(nx - i), \tag{6}$$

Now we can use a Taylor series expansion of $\mathbf{f}(x)$ around x:

$$\mathbf{f}(x) = \sum_{r=0}^{m-1} \frac{1}{r!}\frac{d^r}{dx^r}\mathbf{f}(x)(y-x)^r + \mathbf{R}_m(x,y), \tag{7}$$

where $\mathbf{R}_m(x,y)$ is the Taylor remainder term. We can insert (7) into the formula (6) for the quasiinterpolation operator and obtain

$$Q^n_{\mathbb{R}^K}\mathbf{f}(x) - \mathbf{f}(x) = \sum_{r=1}^{m-1}\frac{1}{r!}\sum_{i=j-R}^{j+R}\sum_{k=-S}^{S}\lambda_k \frac{d^r}{dx^r}\mathbf{f}(x)\left(\frac{i-k}{n}-x\right)^r \Phi(nx-i)$$
$$+ \sum_{i=j-R}^{j+R}\sum_{k=-S}^{S}\lambda_k \mathbf{R}_m(x, \frac{i-k}{n})\Phi(nx-i)$$

By the polynomial generation property (2.1) the first summand in the above formula cancels which yields

$$Q^n_{\mathbb{R}^K}\mathbf{f}(x) - \mathbf{f}(x) = \sum_{i=j-R}^{j+R}\sum_{k=-S}^{S}\lambda_k \mathbf{R}_m(x, \frac{i-k}{n})\Phi(nx-i). \tag{8}$$

For $|k| \leq S$ and $|i-j| \leq R$ we can bound the remainder term as follows

$$\begin{aligned} \left|\mathbf{R}_m(x, \frac{i-k}{n})\right| &\lesssim \left|\left(x - \frac{i-k}{n}\right)^m \frac{d^m}{dx^m}\mathbf{f}(x)\right| \\ &\lesssim n^{-m}\left|\frac{d^m}{dx^m}\mathbf{f}(x)\right|, \end{aligned}$$

which together with (8) yields the desired result

$$\begin{aligned} \|\mathbf{f} - Q^n_{\mathbb{R}^K}\mathbf{f}\|_{L^p} &\lesssim \left(\sum_{k=-S}^{S}|\lambda_k|\right)\left(\sum_{i\in\mathbb{Z}}|\Phi(nx-i)|\right) n^{-m}\left\|\frac{d^m}{dx^m}\mathbf{f}(x)\right\|_{L^p} \\ &\lesssim n^{-m}\|\mathbf{f}\|_{W^{m,p}}. \end{aligned}$$

□

3 Nonlinear Quasiinterpolation

We now aim to generalize the results on quasiinterpolation of Section 2.2.2 to functions $\mathbf{f} : \mathbb{T} \to M$.

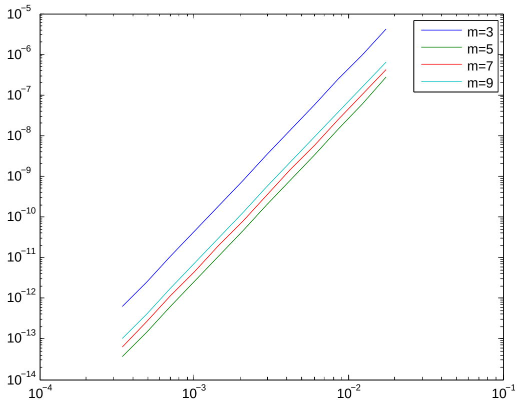
The first issue is to find suitable generalizations of the functions spaces $V^n_{\mathbb{R}^K}$ as defined in Definition 2.5. To this end we present the following definition

Definition 3.1. *We define define the M-valued approximation spaces as*

$$V^n_M := \left\{ \mathbf{P}_M\left(\sum_{i\in[n]} c_i \Phi_n^{PER}(x - i/n)\right) \,:\, (c_i)_{i\in[n]} \subset M \right\}. \tag{9}$$

It is clear that this definition boils down to the usual one in the linear case. A much more subtle problem is whether these spaces V^n_M possess the same approximation properties as in the linear case and to construct quasiinterpolation operators which realize this optimal approximation rate.

It is not at all obvious how to achieve this. Indeed, in Section 3.1 we present a straight forward generalization for quasiinterpolation with manifold-valued data. However this approximation will not have optimal convergence order. The problem to construct optimal quasiinterpolation operators turns out to be surprisingly subtle and requires us to introduce several new tools. Most importantly in Section 3.2 we define the main contribution of the present paper, so-called normal perturbation families. As we shall see later, their existence will allow us to build suitable generalizations of quasiinterpolants also for manifold-valued data. The (rather involved) proof that normal perturbation families actually exist will be the topic of Section 4.



k	Observed convergence rate
3	4.0057
5	4.0037
7	4.0024
9	4.0015

Figure 1: Approximation Order of 'Naive' Generalization

3.1 'Naive' Generalization

A straight forward generalization of quasiinterpolation to manifold-valued functions is

$$QN_M^n(\mathbf{f})(x) := \mathbf{P}_M\left(\sum_{i\in[n]} \mathbf{P}_M\left(\mathrm{circ}_\Lambda \mathfrak{f}_i^n\right)\phi_n^{\mathrm{PER}}(x - i/n)\right).$$

Figure 1 shows measured approximation orders for the function $\mathbf{f}\colon \mathbb{T} \to S^3$ defined by

$$\mathbf{f}(t) = \begin{pmatrix} \sin(x(t))\sin(y(t))\sin(z(t)) \\ \sin(x(t))\sin(y(t))\cos(z(t)) \\ \sin(x(t))\cos(y(t)) \\ \cos(x(t)) \end{pmatrix}$$

where $x(t) = \sin(2t\pi)$, $y(t) = \cos(2t\pi)$ and $z(t) = \sin(2t\pi)$. It suggest that the 'naive' generalization of quasiinterpolation gives a 4-th order approximation rate in contrast to m-th order in the linear case. The next theorem shows that

$$QN_M^n$$

is at least of order 4.

Theorem 3.2. *Let $f \in W^{4,p}(\mathbb{T}, M)$, $l \leq 4$, $\Phi \in W^{l,\infty}$ and $d \geq 4 - l$. Then*

$$\|\mathbf{f} - QN_M^n \mathbf{f}\|_{W^{l,p}} \lesssim n^{l-4}.$$

The following lemma will play an important role not only to prove the previous theorem but also in Chapter 3.2.

Lemma 3.3. *Let m be a positive integer, $U \subset \mathbb{R}^K$ and $\mathbf{f}, (\mathbf{g}_n)_{n\in\mathbb{N}} \in W^{m,p}(\mathbb{T}, U)$ and $\mathbf{P} \in W^{m+1,p}(U, U)$. Furthermore*

$$\sup_n \|\mathbf{g}_n\|_{W^{m,p}} < \infty. \tag{10}$$

Then

$$\|\mathbf{P}(\mathbf{f}) - \mathbf{P}(\mathbf{g}_n)\|_{W^{m,p}} \lesssim \|\mathbf{f} - \mathbf{g}_n\|_{W^{m,p}}.$$

Proof. We prove the theorem by induction on m. For $m = 1$ the statements follows from the chain rule and Lipschitz-continuity of $\mathbf{P}$. From now on lets assume $m > 1$. By the induction hypothesis it is enough to show that

$$\|\frac{d^m}{dx^m}\left(\mathbf{P}(\mathbf{f}) - \mathbf{P}(\mathbf{g}_n)\right)\|_{L^p} \lesssim \|\mathbf{f} - \mathbf{g}_n\|_{W^{m,p}}.$$

The expression $\frac{d^m}{dx^m}(\mathbf{P}(\mathbf{f}(x)) - \mathbf{P}(\mathbf{g}_n(x)))$ is a sum of terms of the form

$$(d^k\mathbf{P})(\mathbf{f}(x))(\frac{d^{a_1}}{dx^{a_1}}\mathbf{f}(x),\dots,\frac{d^{a_k}}{dx^{a_k}}\mathbf{f}(x)) - (d^k\mathbf{P})(\mathbf{g}_n(x))(\frac{d^{a_1}}{dx^{a_1}}\mathbf{g}_n(x),\dots,\frac{d^{a_k}}{dx^{a_k}}\mathbf{g}_n(x)). \tag{11}$$

with $k \leq m$, $a_i \in \mathbb{N}\backslash\{0\}$ and $a_1 + \cdots + a_k = m$. The term (11) can be written as

$$(d^k\mathbf{P})(\mathbf{f}(x))(\frac{d^{a_1}}{dx^{a_1}}\mathbf{f}(x),\dots,\frac{d^{a_k}}{dx^{a_k}}\mathbf{f}(x)) - (d^k\mathbf{P})(\mathbf{g}_n(x))(\frac{d^{a_1}}{dx^{a_1}}\mathbf{f}(x),\dots,\frac{d^{a_k}}{dx^{a_k}}\mathbf{f}(x)) + \tag{12}$$

$$\sum_{i=1}^{k}(d^k\mathbf{P})(\mathbf{g}_n(x))(\frac{d^{a_1}}{dx^{a_1}}\mathbf{g}_n(x),\dots,\frac{d^{a_{i-1}}}{dx^{a_{i-1}}}\mathbf{g}_n(x),\frac{d^{a_i}}{dx^{a_i}}(\mathbf{f}(x)-\mathbf{g}_n(x)),\frac{d^{a_{i+1}}}{dx^{a_{i+1}}}\mathbf{f}(x),\dots,\frac{d^{a_k}}{dx^{a_k}}\mathbf{f}(x)). \tag{13}$$

If $k > 1$ we have $a_j < m$ for all $j \in \{1, 2 \dots, k\}$ and therefore

$$\|\frac{d}{dx^{a_j}}\mathbf{g}_n(x)\|_{L^\infty} \lesssim \|\mathbf{g}_n(x)\|_{W^{m,p}} \lesssim 1.$$

It follows that the L^p-norm of the term can be bounded by $\|\mathbf{f} - \mathbf{g}_n\|_{L^p} + \sum_{i=1}^k \|\mathbf{f} - \mathbf{g}_n\|_{W^{a_i,p}} \lesssim \|\mathbf{f} - \mathbf{g}_n\|_{W^{m-1,p}}$ If $k = 1$ we can bound the term by $\|\mathbf{f} - \mathbf{g}_n\|_{L^p} + \|\mathbf{f} - \mathbf{g}_n\|_{W^{m,p}}$. □

Proof of Theorem 3.2. We first show that

$$|\mathbf{P}_M(\mathrm{circ}_\Lambda \mathfrak{f}_i^n) - \mathrm{circ}_\Lambda \mathfrak{f}_i^n| \lesssim n^{-4}. \tag{14}$$

For any $x, y \in \mathbb{T}$ we can write

$$\mathbf{f}(x+y) = \mathbf{f}(x) + \mathbf{u}(x,y) + \mathbf{v}(x,y)$$

with $\mathbf{u}(x,y) \in TM_{f(x)}$, the tangent space of M at $\mathbf{f}(x)$, and $\mathbf{u}^T(x,y)\mathbf{v}(x,y) = 0$ for all $x, y \in \mathbb{T}$. If $|y| \lesssim n^{-1}$ we have the bounds

$$|\mathbf{u}(x,y)| \lesssim n^{-1}\|\mathbf{f}\|_{W^{1,\infty}} \qquad \text{and} \qquad |\mathbf{v}(x,y)| \lesssim n^{-2}\|\mathbf{f}\|_{W^{1,\infty}}.$$

Let

$$\mathbf{r}_{n,i} = \mathbf{f}(\frac{i}{n}) + \sum_{k=-S}^{S}\lambda_k \mathbf{u}(\frac{i}{n},\frac{i-k}{n}).$$

Since Λ is exact for polynomials of degree 1 we have $|r_{n,i} - \mathbf{f}(\frac{i}{n})| \lesssim n^{-2}$. Note that $u(x,\cdot), v(x,\cdot) \in W^{4,p}(\mathbb{T}, M)$. Furthermore $r_{n,i} \in TM_{f(x)}$ which yields

$$|\mathbf{P}_M(\mathbf{r}_{n,i}) - \mathbf{r}_{n,i}| \lesssim |\mathbf{r}_{n,i}|^2 \lesssim n^{-4}.$$

Hence it follows by using Lemma 3.3 that

$$\begin{aligned}
|\mathbf{P}_M(\mathrm{circ}_\Lambda \mathfrak{f}_i^n) - \mathrm{circ}_\Lambda \mathfrak{f}_i^n| &\leq |\mathbf{P}_M(\mathrm{circ}_\Lambda \mathfrak{f}_i^n) - \mathbf{P}_M(\mathbf{r}_{n,i})| + |\mathbf{P}_M(\mathbf{r}_{n,i}) - \mathbf{r}_{n,i}| && (15)\\
&\quad + \left|\sum_{k=-S}^{S}\lambda_k \mathbf{v}(\frac{i}{n},\frac{i-k}{n})\right| && (16)\\
&\lesssim n^{-4} && (17)\\
& && (18)
\end{aligned}$$

Lemma 3.3 and Theorem 2.6 show that

$$\begin{aligned}
\|\mathbf{f} - QN_M^n\mathbf{f}\|_{W^{l,p}} &\leq \|\mathbf{P}_M(\mathbf{f}) - \mathbf{P}_M(Q_{\mathbb{R}^K}^n\mathbf{f})\|_{W^{l,p}} && (19)\\
&\quad + \left\|\mathbf{P}_M(Q_{\mathbb{R}^K}^n\mathbf{f}) - \mathbf{P}_M\left(\sum_{i\in[n]}\phi_n^{\mathrm{PER}}(x-i/n)\mathbf{P}_M(\mathrm{circ}_\Lambda \mathfrak{f}_i^n)\right)\right\|_{W^{l,p}} && (20)\\
&\lesssim \|\mathbf{f} - Q_{\mathbb{R}^K}^n\mathbf{f}\|_{W^{l,p}} + \left\|\left(\sum_{i\in[n]}\phi_n^{\mathrm{PER}}(x-i/n)\left(\mathbf{P}_M(\mathrm{circ}_\Lambda \mathfrak{f}_i^n) - \mathrm{circ}_\Lambda \mathfrak{f}_i^n\right)\right)\right\|_{W^{l,p}} && (21)\\
&\lesssim n^{l-4}. && (22)
\end{aligned}$$

□

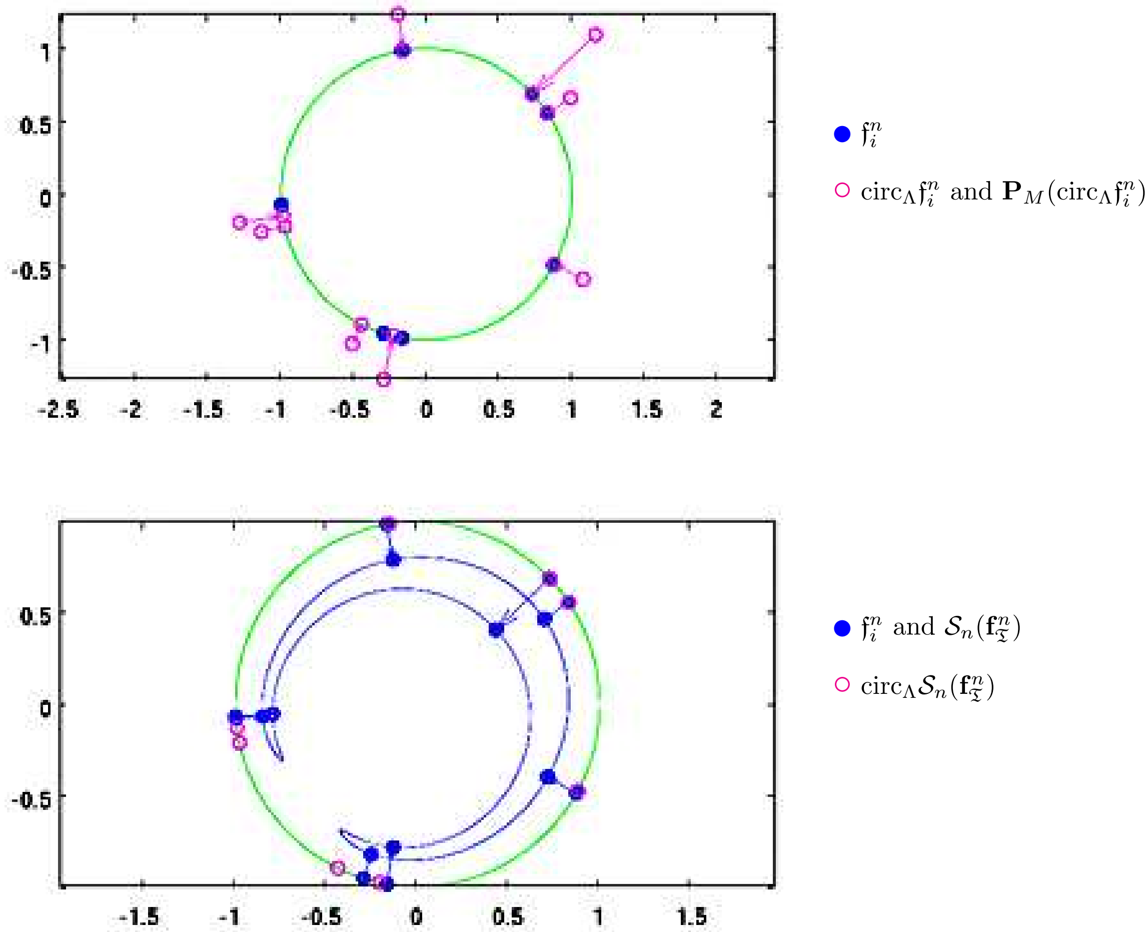


Figure 2: We approximate the function $\mathbf{f}\colon \mathbb{T} \to S^1$ defined by $\mathbf{f}(t) = (\cos(w(t)), \sin(w(t)))^T$ with $w(t) = 2\sin(2t\pi) + 0.8\cos(2t\pi + 2)$. In the upper plot the naive quasiinterpolation from Section 3.1 is shown. Since in general $\mathrm{circ}_\Lambda \mathfrak{f}_i^n \notin M$ we need to project onto the sphere. Due to the projection we get an error of order 4. In the bottom plot we first perturb the sample data orthogonal to the manifold $\mathbf{f}_{\mathfrak{T}}^n$ such that $\mathrm{circ}_\Lambda \mathcal{S}_n(\mathbf{f}_{\mathfrak{T}}^n) \in M$. The projection becomes obsolete and does not contribute an additional error.

3.2 Nonlinear Quasiinterpolation With Normal Perturbations

Our goal is to get approximations with optimal convergence order. To overcome the error due to projection in the last chapter we perturb the data orthogonal to the manifold (see Figure 2).

Definition 3.4. *For* $\mathbf{f} \in C(\mathbb{T}, M)$ *we call a family* $\mathfrak{T} := (\mathbf{t}^n)_{n\in\mathbb{N}} \subset C(\mathbb{T}, \mathbb{R}^D)$ *a* normal perturbation family *for* $\mathbf{f}$ *and* Λ *if*

$$circ_\Lambda \mathcal{S}_n(\mathbf{f}_{\mathfrak{T}}^n) \subset M,$$

where

$$\mathbf{f}_{\mathfrak{T}}^n(x) := \mathbf{f}(x) + \mathbf{N}(\mathbf{f}(x))\mathbf{t}^n(x)$$

Going back to Definition 2.5 we see that a normal perturbation seeks to perturb the function $\mathbf{f}$ in a direction orthogonal to M such that the linear quasiinterpolation operator, applied to the perturbed function, produces control points which lie in M. This is formalized in the following lemma.

Lemma 3.5. *Assume that* $(\mathbf{t}^n)_{n\in\mathbb{N}}$ *is a perturbation family for* $\mathbf{f} \in C(\mathbb{T}, M)$ *and* Λ*. Then for every* $n \in \mathbb{N}$ *we have that*

$$Q^n_M \mathbf{f} := \mathbf{P}_M \left(Q^n_{\mathbb{R}^K} \mathbf{f}^n_{\mathfrak{T}} \right) \in V^n_M. \tag{23}$$

Proof. This is an immediate consequence of the definition of a perturbation family. □

The previous lemma states that, given a normal perturbation family, we may construct an M-valued quasiinterpolation operator by applying a linear quasiinterpolation operator to the perturbed function $\mathbf{f}^n_{\mathfrak{T}}$, followed by a closest-point-projection onto M. In order to carry out an error analysis we will use the fact that by Theorem 2.6 the function $Q^n_{\mathbb{R}^K}\mathbf{f}^n_{\mathfrak{T}}$ is a good approximation of the function $\mathbf{f}^n_{\mathfrak{T}}$. This result requires that the function $\mathbf{f}^n_{\mathfrak{T}}$ is smooth which is formalized in the next definition.

Definition 3.6. *A perturbation family is called* smooth*, if the functions* $\mathbf{t}^n$ *are as smooth as* $\mathbf{f}$*, independent of* n*, more precisely*

$$\|\mathbf{t}^n\|_{W^{m,p}} \lesssim 1$$

with the implicit constant independent of n *and only depending on* $\|\mathbf{f}\|_{W^{m,p}}$*.*

Now we are ready to state and prove our first main result, namely that if we assume the existence of a smooth normal perturbation family, we get optimal approximation orders using the manifold-valued quasiinterpolation operators Q^n_M.

Proposition 3.7. *Let* $m > \frac{1}{p}$*,* $\mathbf{f} \in W^{m,p}$ *and assume that there exists a smooth perturbation family* $\mathfrak{T}$ *for* $\mathbf{f}$ *and* Λ *then*

$$\|\mathbf{f} - Q^n_M \mathbf{f}\|_{W^{l,p}} \lesssim n^{l-m},$$

whenever $d \geq m - l$ *and* $\Phi \in W^{\infty,l}$*.*

Proof. We use Theorem 2.6 for the functions $\mathbf{g}_n(x) = \mathbf{f}(x) + \mathbf{N}(\mathbf{f}(x))\mathbf{t}^n(x)$ and Lemma 3.3 to obtain

$$\begin{aligned} \|\mathbf{f} - Q^n_M \mathbf{f}\|_{W^{l,p}} &= \|\mathbf{P}_M(\mathbf{g}_n) - \mathbf{P}_M(Q^n_{\mathbb{R}^K}\mathbf{g}_n)\|_{W^{l,p}} && (24)\\ &\lesssim \|\mathbf{g}_n - Q^n_{\mathbb{R}^K}\mathbf{g}_n\|_{W^{l,p}} && (25)\\ &\lesssim n^{l-m}\|\mathbf{g_n}\|_{W^{m,p}}. && (26) \end{aligned}$$

We have

$$\begin{aligned} \|\mathbf{g_n}\|_{W^{m,p}} &\lesssim \|\mathbf{f}\|_{W^{m,p}} + \|\mathbf{N}(\mathbf{f})\|_{L^\infty}\|\mathbf{t}^n\|_{W^{m,p}} + \sum_{i=1}^{m} \|\mathbf{N}(\mathbf{f})\|_{W^{i,p}}\|\mathbf{t}^n\|_{W^{m-i,\infty}} && (27)\\ & && (28) \end{aligned}$$

The expression $\frac{d^i}{dx^i}\mathbf{N}(\mathbf{f}(x))$ is a sum of terms of the form

$$(d^k\mathbf{N})(\mathbf{f}(x))\left(\frac{d^{a_1}}{dx^{a_1}}\mathbf{f}(x), \ldots, \frac{d^{a_k}}{dx^{a_k}}\mathbf{f}(x)\right) \tag{29}$$

with $k \leq i$, $a_j \in \mathbb{N}\backslash\{0\}$ and $a_1 + \cdots + a_k = i$.
As

$$\left\|\frac{d^{a_j}}{dx^{a_j}}\mathbf{f}\right\|_{L^p} \lesssim \|\mathbf{f}\|_{W_{i,p}} \lesssim 1 \tag{30}$$

it follows that

$$\|\mathbf{N}(\mathbf{f})\|_{W^{i,p}} \lesssim 1.$$

Since the perturbation family $\mathfrak{T}$ is smooth we have

$$\|\mathbf{t}^n\|_{W^{m,p}} \lesssim 1.$$

and because of (30) we have for $i \geq 1$

$$\|\mathbf{t}^n\|_{W^{m-i,\infty}} \lesssim 1.$$

Hence we get $\|\mathbf{g_n}\|_{W^{m,p}} \lesssim 1$ which finishes our proof.

□

In the following section we will see that for each $\mathbf{f}$ there exists essentially a unique perturbation family. From this result and Proposition 3.7 we will be able to construct optimal quasiinterpolation operators for the spaces V_M^n.

Remark 3.8. *It should be clear that the general results of this section hold for any family of approximation spaces which are defined by constraining the control points to M and applying a closest-point projection. In this way potentially any linear approximation result can be generalized to the manifold-valued setting, provided that one can establish the existence of smooth normal perturbation families. Establishing this latter fact in a more general setting will be the subject of future work.*

4 Construction of Normal Perturbation Families

From Section 3.2 it follows that in order to construct optimal quasiinterpolation operators onto the spaces V_M^n, it is sufficient to establish the existence of smooth normal perturbation families. This is the task of the present section where we establish this under the following natural positivity assumption on the symbol of Λ.

Assumption 4.1. *We consider a convolution mask $\Lambda = (\lambda_j)_{j=-S}^S \subset \mathbb{R}$, such that*

$$\sum_{j=-S}^{S} \lambda_j = 1 \quad \text{and} \tag{31a}$$

$$\hat{\Lambda}(\omega) := \sum_{j=-S}^{S} \lambda_j \exp(2\pi i j \omega) > 0 \quad \text{for all } \omega \in \mathbb{T}. \tag{31b}$$

It is well-known that under Assumption 4.1, the corresponding circular convolution operator and its inverse are bounded operators.

Proposition 4.2. *For all $p \in [1, \infty]$ and n, $K \in \mathbb{N}$, whenever Assumption 4.1 holds, the circular convolution satisfies*

$$\|circ_\Lambda\|_{\ell^p([n],\mathbb{R}^K)\to\ell^p([n],\mathbb{R}^K)} < \infty \quad \text{and} \quad \|circ_\Lambda^{-1}\|_{\ell^p([n],\mathbb{R}^K)\to\ell^p([n],\mathbb{R}^K)} < \infty. \tag{32}$$

The implicit constant is independent of $n \in \mathbb{N}$.

Proof. We restrict ourselves to the case $K = 1$. The generalization to $K > 1$ is straight forward. Define the cyclic matrix $\mathbf{A}_{n,\lambda} \in \mathbb{R}^{n,n}$ by

$$(\mathbf{A}_{n,\lambda})_{ij} = a_{ij}^{n,\lambda} = \begin{cases} \lambda_{i-j \mod n} & |i-j \mod n| \leq S \\ 0 & \text{otherwise.} \end{cases}.$$

We denote the matrix we get by replacing each entry of a Matrix $\mathbf{M}$ by its absolute value by $|\mathbf{M}|$. As $\|\mathrm{circ}_\Lambda\|_{\ell^p([n],\mathbb{R}^K)\to\ell^p([n],\mathbb{R}^K)} = \||\mathbf{A}_{n,\lambda}|\|_p$ and $\|\mathrm{circ}_\Lambda^{-1}\|_{\ell^p([n],\mathbb{R}^K)\to\ell^p([n],\mathbb{R}^K)} = \||\mathbf{A}_{n,\lambda}^{-1}|\|_p$ it is enough to show that $\||\mathbf{A}_{n,\lambda}|\|_p$ and $\||\mathbf{A}_{n,\lambda}^{-1}|\|_p$ are bounded independent of n. Note that

$$\||\mathbf{A}_{n,\lambda}|\|_1 = \||\mathbf{A}_{n,\lambda}|\|_\infty = \sum_{i=-S}^{S} |\lambda_j|.$$

By the Riesz-Thorin theorem [19] it follows that $\||\mathbf{A}_{n,\lambda}|\|_p \leq \sum_{i=-S}^S |\lambda_j|$ and by the lemma of Wiener [21] there exists a sequence $(b_j)_{j\in\mathbb{Z}}$ such that

$$\sum_{j\in\mathbb{Z}} b_j \exp(2\pi i j\omega) = \frac{1}{\hat{\Lambda}(\omega)}$$

and $\sum_{j\in\mathbb{Z}} |b_j| < \infty$. The inverse $\mathbf{A}_{n,\lambda}^{-1}$ of $\mathbf{A}_{n,\lambda}$ can be written as

$$(\mathbf{A}_{n,\lambda}^{-1})_{ij} = \sum_{k\in Z} b_{i-j+nk}.$$

It follows that

$$|||\mathbf{A}_{n,\lambda}^{-1}|||_\infty = |||\mathbf{A}_{n,\lambda}^{-1}|||_1 \leq \sum_{k\in\mathbb{Z}} |b_k|.$$

Again by the Riesz-Thorin theorem it follows that $|||\mathbf{A}_{n,\lambda}^{-1}|||_p \leq \sum_{k\in\mathbb{Z}} |b_k|$.

□

4.1 Main Result

The purpose of this subsection is to establish the existence of smooth normal perturbation families for any function $\mathbf{f} \in W^{m,p}(\mathbb{T}, M)$ with $m > 1/p$. We will make use of the following definition.

Definition 4.3. *For $\varepsilon > 0$ and for $\Lambda = (\lambda_i)_{i=-S}^S$ define the function $\mathfrak{F}^{\Lambda,n,\varepsilon} : \ell\left([n], \mathbb{R}^D\right) \times \ell\left([n], \mathbb{R}^K\right) \to \ell\left([n], \mathbb{R}^D\right)$ by*

$$\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z}, \mathfrak{f})_i := \mathfrak{F}_i^{n,\varepsilon}(\mathfrak{z}, \mathfrak{f}) := \mathbf{g}\left(\varepsilon \mathit{circ}_\Lambda\left(\mathfrak{f}\right)_i + (1-\varepsilon)\mathbf{f}_i + \mathit{circ}_\Lambda\left(\mathbf{N}(\mathfrak{f})\mathfrak{z}\right)_i\right), \tag{33}$$

where we write

$$\left(\mathbf{N}(\mathfrak{f})\mathfrak{z}\right)_i = \mathbf{N}(\mathbf{f}_i)\mathbf{z_i} \in \mathbb{R}^K.$$

Definition 3.4 essentially states that the defining condition for a normal perturbation family is that

$$\mathfrak{F}^{\Lambda,n,1}(\mathcal{S}_n \mathbf{t}^n, \mathcal{S}_n \mathbf{f}) = \mathfrak{o}, \tag{34}$$

where $\mathfrak{o}$ is the zero sequence with $\mathfrak{o}_i = \mathbf{0}$ for all $i \in [n]$. Therefore in the following we will examine the existence and uniqueness of solutions to (34), given the function $\mathbf{f}$. The parameter ε will be needed in Theorem 4.8 to show the existence of solutions to (34) which will be achieved using a topological argument where we deform the mapping $\mathfrak{F}^{\Lambda,n,0}$ to the mapping $\mathfrak{F}^{\Lambda,n,1}$.

Denote

$$D_\mathfrak{z}\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z}, \mathfrak{f}) : \ell\left([n], \mathbb{R}^D\right) \to \ell\left([n], \mathbb{R}^D\right)$$

the Jacobian of $\mathfrak{F}^{\Lambda,n,\varepsilon}$ in the variable $\mathfrak{z}$, i.e.,

$$D_\mathfrak{z}\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z}, \mathfrak{f})_{i,j} = \frac{\partial}{\partial \mathbf{z}_j}\mathfrak{F}_i^{n,\varepsilon}(\mathfrak{z}, \mathfrak{f}) \in \mathbb{R}^{D\times D}. \tag{35}$$

Our first result is that the derivative of $\mathfrak{F}^{\Lambda,n,\varepsilon}$ is given, up to a controlled error, by a circular convolution with Λ.

Proposition 4.4. *We have*

$$D_\mathfrak{z}\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z}, \mathfrak{f})_{i,j} = \begin{cases} \lambda_{i-j}\mathbf{I}_D + \mathcal{O}\left(\|\nabla\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K)} + \|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)}\right) & j \in i + [-S, S] \\ \mathbf{0} \in \mathbb{R}^{D\times D} & \mathit{else} \end{cases} \tag{36}$$

with the implicit constant independent of n.

Proof. We have for $j \in i + [-S, S]$ that

$$D_\mathfrak{z}\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z}, \mathfrak{f})_{i,j} = D\mathbf{g}\left(\varepsilon \mathrm{circ}_\Lambda\left(\mathfrak{f}\right)_i + (1-\varepsilon)\mathbf{f}_i + \mathrm{circ}_\Lambda\left(\mathbf{N}(\mathfrak{f})\mathfrak{z}\right)_i\right)^T \lambda_{i-j}\mathbf{N}(\mathbf{f}_j). \tag{37}$$

Further, by (31a), we can estimate

$$\begin{aligned} \left|\varepsilon \mathrm{circ}_\Lambda\left(\mathfrak{f}\right)_i + (1-\varepsilon)\mathbf{f}_i - \mathbf{f}_j\right| &\leq \left|\varepsilon \sum_{k=-S}^{S} \lambda_k\left(\mathbf{f}_{i-k} - \mathbf{f}_j\right) + (1-\varepsilon)\left(\mathbf{f}_i - \mathbf{f}_j\right)\right| \\ &\leq 2S \sum_{l=-S}^{S} |\lambda_l| \max_{k\in i+[-S,S]} |\nabla\mathfrak{f}_k| \\ &\lesssim \|\nabla\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K)}, \end{aligned}$$

as well as

$$\begin{aligned} \left|\mathrm{circ}_\Lambda\left(\mathbf{N}(\mathfrak{f})\mathfrak{z}\right)_i\right| &= \left|\sum_{k=-S}^{S}\lambda_k\mathbf{N}(\mathbf{f}_{i-k})\mathbf{z}_{i-k}\right| \lesssim \max_{k\in i+[-S,S]}|\mathbf{z}_k| \\ &\lesssim \|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)}. \end{aligned}$$

Therefore we have the estimate

$$\left|\mathrm{circ}_\Lambda\left(\mathfrak{f}\right)_i + (1-\varepsilon)\mathbf{f}_i + \mathrm{circ}_\Lambda\left(\mathbf{N}(\mathfrak{f})\mathfrak{z}\right)_i - \mathbf{f}_j\right| \lesssim \|\nabla\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K} + \|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)}. \tag{38}$$

Combining (37) and (38) we get that

$$D_\mathfrak{z}\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z},\mathfrak{f})_{i,j} = \lambda_{i-j}D\mathbf{g}(\mathbf{f}_j)\mathbf{N}(\mathbf{f}_j) + \mathcal{O}\left(\|\nabla\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K} + \|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)}\right),$$

which together with (2) yields the desired result. □

The previous proposition states that the Jacobian of the function $\mathfrak{F}$ is, up to a small perturbation by a banded matrix, given by circ_Λ:

Corollary 4.5. *We have*

$$D_\mathfrak{z}\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z},\mathfrak{f}) = circ_\Lambda + \mathcal{A}(\mathfrak{z},\mathfrak{f}), \tag{39}$$

where for every $p\in[1,\infty]$ it holds that

$$\|\mathcal{A}(\mathfrak{z},\mathfrak{f})\|_{\ell^p([n],\mathbb{R}^D)\to\ell^p([n],\mathbb{R}^D)} = \mathcal{O}\left(\|\nabla\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K} + \|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)}\right) \tag{40}$$

with the implicit constant independent of n.

We can now state the following injectivity result for the function $\mathfrak{F}$:

Proposition 4.6. *Fix $C,m>0$, $0<\nu_0<\frac{1}{2}\left(\sup_{n\in\mathbb{N}}\|circ_\Lambda\|^{-1}_{\ell^p([n],\mathbb{R}^K)\to\ell^p([n],\mathbb{R}^K)}\right)^{-1}$ and $p\in[1,\infty]$. There exists $n_0\in\mathbb{N}$ such that for all $n>n_0$, and $\mathfrak{z}$, $\mathfrak{z}'$, $\mathfrak{f}$ with*

$$\|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)}, \|\mathfrak{z}'\|_{\ell^p([n],\mathbb{R}^D)} \le \nu_0, \quad \|\mathfrak{f}\|_{\ell^{m,p}([n],\mathbb{R}^K)} \le C$$

and $\varepsilon\in\mathbb{T}$ we have

$$\left\|\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z},\mathfrak{f}) - \mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z}',\mathfrak{f})\right\|_{\ell^p([n],\mathbb{R}^D)} \gtrsim \|\mathfrak{z}-\mathfrak{z}'\|_{\ell^p([n],\mathbb{R}^D)}. \tag{41}$$

Proof. Follows from Corollary 4.5 together with the fact that by (31b) and Corollary 4.2 the operator circ_Λ has a bounded inverse in $\ell^p\left([n],\mathbb{R}^D\right)$ with bound independent of n: We have the representation

$$\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z},\mathfrak{f}) - \mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z}',\mathfrak{f}) = \int_0^1 D_\mathfrak{z}\mathfrak{F}^{\Lambda,n,\varepsilon}(t\mathfrak{z}+(1-t)\mathfrak{z}',\mathfrak{f})dt(\mathfrak{z}-\mathfrak{z}'), \tag{42}$$

which by Corollary 4.5 can be written as

$$\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z},\mathfrak{f}) - \mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z}',\mathfrak{f}) = \left(\mathrm{circ}_\Lambda + \mathcal{B}\right)(\mathfrak{z}-\mathfrak{z}'), \tag{43}$$

for some $\mathcal{B}$ which satisfies the bounds

$$\|\mathcal{B}\|_{\ell^p([n],\mathbb{R}^D)\to\ell^p([n],\mathbb{R}^D)} = \mathcal{O}\left(\|\nabla\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K)} + \max(\|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)}, \|\mathfrak{z}'\|_{\ell^p([n],\mathbb{R}^D)})\right) \tag{44}$$

with the implicit constant independent of n. It follows from (43) and (44) that

$$\begin{aligned} \left\|\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z},\mathfrak{f}) - \mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z}',\mathfrak{f})\right\|_{\ell^p([n],\mathbb{R}^D)} &\gtrsim \|\mathrm{circ}_\Lambda(\mathfrak{z}-\mathfrak{z}')\|_{\ell^p([n],\mathbb{R}^D)} \\ &\quad - \left(\max(\|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)}, \|\mathfrak{z}'\|_{\ell^p([n],\mathbb{R}^D)}) + \|\nabla\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K)}\right)\|\mathfrak{z}-\mathfrak{z}'\|_{\ell^p([n],\mathbb{R}^D)} \\ &\ge \left(\frac{1}{2}\left\|\mathrm{circ}_\Lambda^{-1}\right\|^{-1}_{\ell^p([n],\mathbb{R}^D)\to\ell^p([n],\mathbb{R}^D)} - n^{-\min(m,1)}C\right)\|\mathfrak{z}-\mathfrak{z}'\|_{\ell^p([n],\mathbb{R}^D)}. \end{aligned}$$

Picking n large enough yields the desired claim. □

Lemma 4.7. *We have for all $\varepsilon, \varepsilon' \in \mathbb{T}$ and $p \in [1,\infty]$ the estimate*

$$\left\|\mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z},\mathfrak{f}) - \mathfrak{F}^{\Lambda,n,\varepsilon'}(\mathfrak{z},\mathfrak{f})\right\|_{\ell^p([n],\mathbb{R}^D)} \lesssim |\varepsilon - \varepsilon'| \|\nabla \mathfrak{f}\|_{\ell^p([n],\mathbb{R}^K)},$$

with the implicit constant independent of $\mathfrak{z}$ and n.

Proof. This follows directly from the Lipschitz-continuity of $\mathbf{g}$ and (31a). □

Now we are ready to prove the first main result of this section, namely the existence of a unique solution to (34). The uniqueness has already been shown above in Proposition 4.6. In order to show existence of a solution we first show the existence of a solution to the corresponding equation with $\varepsilon = 0$ and use a topological argument to infer existence for $\varepsilon = 1$.

Theorem 4.8. *Fix $C, m > 0$, $0 < \nu_0 < \frac{1}{2}\left(\sup_{n\in\mathbb{N}} \|circ_\Lambda\|^{-1}_{\ell^p([n],\mathbb{R}^K)\to\ell^p([n],\mathbb{R}^K)}\right)^{-1}$ and $p \in [1,\infty]$. Then there exists $n_0 > 0$ such that for all $n > n_0$ and all*

$$\mathfrak{f} \in \ell\left([n],\mathbb{R}^K\right) \quad \text{with} \quad \|\mathfrak{f}\|_{\ell^{m,p}([n],\mathbb{R}^K)} \le C,$$

there exists a unique $\mathfrak{z}_\mathfrak{f} \in \ell\left([n],\mathbb{R}^D\right)$ with

$$\|\mathfrak{z}_\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^D)} \le \nu_0$$

and

$$\mathfrak{F}^{\Lambda,n,1}(\mathfrak{z}_\mathfrak{f},\mathfrak{f}) = \mathfrak{o}.$$

Proof. First we note that

$$\mathfrak{F}^{\Lambda,n,0}(\mathfrak{o},\mathfrak{f}) = \mathfrak{o}. \tag{45}$$

Fix n, ν_0 such that Proposition 4.6 is valid, e.g., (41) holds. Define

$$\varepsilon_* := \inf\{\varepsilon > 0 : \nexists \mathfrak{z} : \|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)} \le \nu_0 \wedge \mathfrak{F}^{\Lambda,n,\varepsilon}(\mathfrak{z},\mathfrak{f}) = \mathfrak{o}\}.$$

Assume that $\varepsilon_* < 1$. For ease of notation in the following we omit the dependence on $\mathfrak{f}$ in our notation for the function $\mathfrak{F}$. Then by Lemma 4.7 we have that

$$\mathfrak{o} \in \partial\left(\mathfrak{F}^{\Lambda,n,\varepsilon_*}\left\{\mathfrak{z} \in \ell([n],\mathbb{R}^D) : \|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)} \le \nu_0\right\}\right).$$

Since by Proposition 4.6, the function $\mathfrak{F}^{\Lambda,n,\varepsilon_*}$ is a homeomorphism, we have that

$$\partial\left(\mathfrak{F}^{\Lambda,n,\varepsilon_*}\left\{\mathfrak{z} \in \ell([n],\mathbb{R}^D) : \|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)} \le \nu_0\right\}\right) = \mathfrak{F}^{\Lambda,n,\varepsilon_*}\partial\left\{\mathfrak{z} \in \ell([n],\mathbb{R}^D) : \|\mathfrak{z}\|_{\ell^p([n],\mathbb{R}^D)} \le \nu_0\right\}$$

and consequently, there exists $\mathfrak{z}_*$ with $\|\mathfrak{z}_*\|_{\ell^p([n],\mathbb{R}^D)} = \nu_0$ and $\mathfrak{F}^{\Lambda,n,\varepsilon_*}(\mathfrak{z}_*) = \mathfrak{o}$. By Proposition 4.6 we have

$$\|\mathfrak{F}^{\Lambda,n,\varepsilon_*}(\mathfrak{o})\|_{\ell^p([n],\mathbb{R}^D)} = \|\mathfrak{F}^{\Lambda,n,\varepsilon_*}(\mathfrak{z}_*) - \mathfrak{F}^{\Lambda,n,\varepsilon_*}(\mathfrak{o})\|_{\ell^p([n],\mathbb{R}^D)} \gtrsim \nu_0, \tag{46}$$

for all $n \in \mathbb{N}$. On the other hand, using Lemma 4.7 and (45), we have that

$$\|\mathfrak{F}^{\Lambda,n,\varepsilon_*}(\mathfrak{o})\|_{\ell^p([n],\mathbb{R}^D)} = \|\mathfrak{F}^{\Lambda,n,\varepsilon_*}(\mathfrak{o}) - \mathfrak{F}^{\Lambda,n,0}(\mathfrak{o})\|_{\ell^p([n],\mathbb{R}^D)} \lesssim \|\nabla\mathfrak{f}\|_{\ell^p([n],\mathbb{R}^D)} \lesssim n^{-\min(m,1)}. \tag{47}$$

Clearly, this becomes a contradiction for n large enough. This shows the existence of $\mathfrak{z}_\mathfrak{f}$. The uniqueness follows from Proposition 4.6. □

Now we shall use Theorem 4.8 to construct smooth normal perturbation families for each function $\mathbf{f} \in W^{m,p}(\mathbb{T}, M)$ if $m > 1/p$. More precisely, putting $\tau_x : f(\cdot) \to f(\cdot - x)$ the shift operator we make the following definition.

Definition 4.9. *For a function $\mathbf{f} \in W^{m,p}(\mathbb{T}, M)$, $m > 1/p$, we define the family $\mathfrak{T} = (\mathbf{t}^n_\mathbf{f})_{n\in\mathbb{N}}$ as*

$$\mathbf{t}^n_\mathbf{f}(x) := \left(\mathfrak{z}_{\mathcal{S}^n(\tau_x \mathbf{f})}\right)_0.$$

It is easy to see that the previous definition yields a normal perturbation family as in Definition 3.4.

Lemma 4.10. *The family* $\mathfrak{T}$ *from Definition 4.9 is a normal perturbation family for* $\mathbf{f}$ *and* Λ*.*

Proof. First, since $\mathbf{f} \in W^{m,p}(\mathbb{T}, M)$ there exists $C > 0$ such that

$$\|\mathcal{S}^n(\tau_x \mathbf{f})\|_{\ell^{\min(1,m),p}([n],\mathbb{R}^K)} < C, \quad \text{for all } x \in \mathbb{T}.$$

Therefore, by Theorem 4.8 there exists n_0 such that for all $n > n_0$, the function $\mathbf{t}_\mathbf{f}^n$ is uniquely defined. The fact that $\mathfrak{T}$ constitutes a normal perturbation family follows directly from Definition 3.4, together with the fact that

$$\mathcal{S}^n(\mathbf{t}_\mathbf{f}^n) = \mathfrak{z}_{\mathcal{S}^n \mathbf{f}}.$$

□

Now that we have shown the existence of an essentially unique normal perturbation family we can study the associated quasiinterpolation operators Q_M^n as introduced in (23). In order to apply Theorem 3.7 to this setting it only remains to show that the normal perturbation family of Definition 4.9 is smooth. This is shown in the following lemma.

Proposition 4.11. *The function* $\mathbf{t}_\mathbf{f}^n$ *satisfies*

$$\|\mathbf{t}_\mathbf{f}^n\|_{W^{k,p}(\mathbb{T},\mathbb{R}^D)} \lesssim 1 \quad \textit{for all} \quad k \leq m$$

with the implicit constant independent of n*. In other words, the family* $\mathfrak{T} := (\mathbf{t}_\mathbf{f}^n)_{n\in\mathbb{N}}$ *is a smooth normal perturbation family for* $\mathbf{f}$*.*

Proof. First we note that we have

$$\mathfrak{o} = \mathfrak{F}^{\Lambda,n,1}\left(\mathcal{S}^n\left(\tau_x \mathbf{t}_\mathbf{f}^n\right), \mathcal{S}^n\left(\tau_x \mathbf{f}\right)\right) \quad \text{for all } x \in \mathbb{T}. \tag{48}$$

Repeatedly using the chain rule on equation (48) will yield the desired claim. We start with the case $k = 0$. Then we have by Theorem 4.8 that

$$\|\mathcal{S}^n\left(\tau_x \mathbf{t}_\mathbf{f}^n\right)\|_{\ell^p([n],\mathbb{R}^D)} \lesssim 1,$$

which implies that

$$\sum_{i\in[n]} \left|\mathbf{t}_\mathbf{f}^n\left(x + \frac{i}{n}\right)\right|^p \lesssim 1 \quad \text{for all} \quad x \in \mathbb{T},$$

which implies that

$$\|\mathbf{t}_\mathbf{f}^n\|_{L^p}^p = \int_0^1 |\mathbf{t}_\mathbf{f}^n(x)|^p \, dx = \int_0^{1/n} \sum_{i\in[n]} \left|\mathbf{t}_\mathbf{f}^n\left(x + \frac{i}{n}\right)\right|^p dx \lesssim 1.$$

Assume from now on, that $k > 0$. Since we have $m > 1/p$ all derivatives of order $\leq m-1$ of f are bounded. The rest of the argument can be shown by induction, noticing that computing the m-th derivative in x of 48 yields the relation

$$\mathfrak{o} = D_\mathfrak{z} \mathfrak{F}^{\Lambda,n,1}\left(\mathcal{S}^n\left(\tau_x \mathbf{t}_\mathbf{f}^n\right), \mathcal{S}^n\left(\tau_x \mathbf{f}\right)\right) \frac{d^m}{dx^m} \mathcal{S}^n\left(\tau_x \mathbf{t}_\mathbf{f}^n\right) + \text{l.o.t},$$

where l.o.t. denotes terms containing lower order derivatives of $\mathcal{S}^n\left(\tau_x \mathbf{t}_\mathbf{f}^n\right)$. The elements of l.o.t. contain products of derivatives of $\mathcal{S}^n\left(\tau_x \mathbf{t}_\mathbf{f}^n\right)$ of orders smaller than m and derivatives of $\mathfrak{F}^{\Lambda,n,1}$ and $\mathcal{S}^n\left(\tau_x \mathbf{f}\right)$ of order smaller or equal to n (this follows directly from the product rule and the chain rule). By the induction hypothesis and $\|u\|_{W^{k,\infty}} \lesssim \|u\|_{W^{m,p}}$ for all $k < m$ and $u \in W^{m,p}$ we can bound l.o.t. by a constant.
Now we note that the derivative

$$D_\mathfrak{z} \mathfrak{F}^{\Lambda,n,1}\left(\mathcal{S}^n\left(\tau_x \mathbf{t}_\mathbf{f}^n\right), \mathcal{S}^n\left(\tau_x \mathbf{f}\right)\right)$$

is, for n sufficiently large, a bounded isomorphism on $\ell^p([n],\mathbb{R}^D)$, a direct consequence of Proposition 4.2 and Corollary 4.5. This, together with $\|\text{l.o.t}\|_{\ell^p([n],\mathbb{R}^D)} \lesssim 1$ yields that

$$\left\| \frac{d^m}{dx^m} \mathcal{S}^n \left(\tau_x \mathbf{t}_{\mathbf{f}}^n\right) \right\|_{\ell^p([n],\mathbb{R}^D)} \lesssim 1. \tag{49}$$

We can now apply the same integration argument as in the case $m=0$ and obtain the desired bound

$$\left\| \frac{d^m}{dx^m} \mathbf{t}_{\mathbf{f}}^n \right\|_{L^p(\mathbb{T},\mathbb{R}^D)} \lesssim 1. \tag{50}$$

□

We can now state our main result.

Theorem 4.12. *Assume that $\Phi \in C(\mathbb{T},\mathbb{R})$ satisfies Assumption 2.1 with a sequence Λ which satisfies Assumption 4.1. For $\mathbf{f} \in W^{m,p}$, $m > 1/p$ and $p \in [1,\infty]$ define the nonlinear quasiinterpolation operator $Q_M^n : C(\mathbb{T}, M) \to V_M^n$ via (23), where $\mathfrak{T}$ is the smooth normal perturbation family for $\mathbf{f}$. Then we have*

$$\|\mathbf{f} - Q_M^n \mathbf{f}\|_{W^{l,p}} \lesssim n^{l-m},$$

whenever $m \leq d$ and $\Phi \in W^{\infty,p}$. In particular the approximation spaces V_M^n realize the optimal approximation rate.

Proof. By Proposition 4.11 the family $\mathfrak{T}$ as defined in Definition 4.9 is a smooth normal perturbation family. Therefore we can apply Proposition 3.7 which yields the desired result. □

4.2 Choice of Λ

From the result in the previous subsections it follows that one can construct optimal M-valued quasiinterpolation operators, provided that there exist basis fuctions Φ and preprocessing sequences Λ such that Assumptions 2.1 and 4.1 are satisfied. It is the purpose of the present subsection to verify that this is indeed the case.

Proposition 4.13. *For $m \in \mathbb{N}$ odd let $\Phi = B_m$ the cardinal B-spline as defined in Example 2.2. Then there exists a sequence Λ such that polynomial reproduction of degree $d = m+1$ as in Assumption 2.1 is satisfied and also Assumption 4.1 holds true.*

Proof. The proof follows directly from results in [3]. For $z \in \mathbb{C}$ define $N_m(z) := \sum_{k=-(m-1)/2}^{(m-1)/2} B_m(k) z^k$, $D(z) = 1 - N_m(z)$ and Λ by

$$\Lambda(z) = \sum_{k=-(m-1)^2/4}^{(m-1)^2/4} \lambda_k z^k = \sum_{l=0}^{(m-1)/2} (D(z))^l.$$

As $0 \leq N_m(z) \leq 1$ for all $z \in \mathbb{C}$ with $\|z\| = 1$ we have

$$\Lambda(z) = \sum_{l=0}^{(m-1)/2} (D(z))^l \geq 1 \quad \text{for all } z \in \mathbb{C} \text{ with } \|z\| = 1.$$

Therefore Assumption 4.1 is satisfied.
The polynomial exactness condition 2.1 can be written as

$$\sum_{i=-(m-1)/2}^{(m-1)/2} B_m(i) \sum_{j=-S}^{S} \lambda_j (i+j)^r = \begin{cases} 1 & r = 0 \\ 0 & r \in \{1,\ldots,m\} \end{cases}$$

which is equivalent to

$$\frac{d^r}{dz^r} \left(N_m(z)\Lambda(z)\right)|_{z=1} = \begin{cases} 1 & r = 0 \\ 0 & r \in \{1,\ldots,m\}. \end{cases}$$

As $D(z)$ is a symmetric Laurent polynomial with 1 as a zero we have $D(z) = \frac{(z-1)^2}{z} q(z)$ with a symmetric Laurent polynomial q and

$$\Lambda(z) N_m(z) = \left(\sum_{l=0}^{(m-1)/2} (D(z))^l \right) (1 - D(z)) = 1 - D(z)^{(m+1)/2} = 1 + \frac{(z-1)^{m+1}}{z^{(m+1)/2}} (q(z))^{(m+1)/2}.$$

Hence it follows that

$$\frac{d^r}{dz^r} \left(N_m(z) \Lambda(z) \right) |_{z=1} = \begin{cases} 1 & r = 0 \\ 0 & r \in \{1, \dots, m\} \end{cases}$$

which shows that Assumption 2.1 holds true. □

Now we have assembled all pieces to prove the main result of this paper.

Theorem 4.14. *For $k \geq m-1$ odd let $\Phi = B_k$, Λ chosen as in Prop 4.13, Q_M^n defined as in Definition 23 and assume that $\mathbf{f} \in W^{m,p}$. Then for every $n \in \mathbb{N}$ we have that*

$$\|\mathbf{f} - Q_M^n \mathbf{f}\|_{W^{l,p}} \lesssim n^{l-m},$$

Proof. This follows directly from Theorem 4.12 and Proposition 4.13. □

The sequence in the proof of Proposition 4.13 has length $\approx m^2/2$. An interesting question is if there are any sequences with shorter length. e.g. $\approx m$. For each odd $m \in \mathbb{N}$ there exists a sequence of length m such that the polynomial reproduction property with Φ a B-spline of degree m holds. For small m these sequences also satisfy the positive Fouriertransform condition as Figure 4.2 shows. However the authors did not succeed to prove it for general m.

It is also unclear if Assumption 4.1 is necessary. Numerical experiments suggest that we get optimal

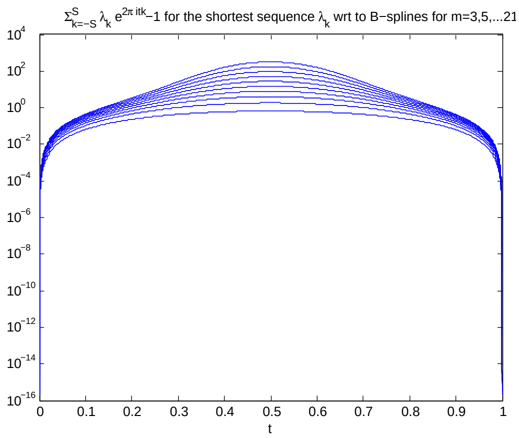
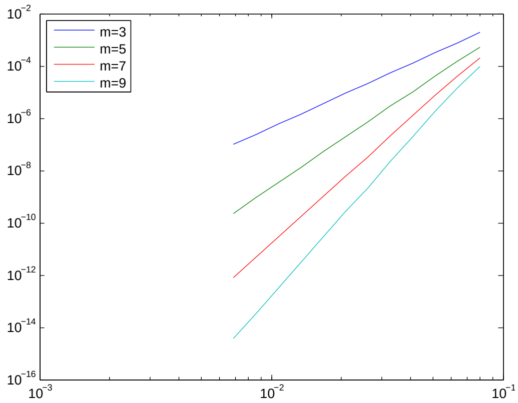


Figure 3: Our conjecture is that the Fouriertransform is always larger or equal to 1. The lowest curve corresponds to $m = 3$ and it seems that the family of the Fouriertransforms is monotonically increasing in m.

convergence order even if Assumption 4.1 is not satisfied. As an example we can consider

$$(0, 13/240, -7/15, 73/40, -7/15, 13/240, 0) + c \cdot (1, -6, 15, -20, 15, -6, 1).$$

For any $c \in \mathbb{R}$ the sequence together with quintic b-splines satisfies the polynomial reproduction assumption for polynomials of degree 5. However Assumption 4.1 is not satisfied if e.g. $c = 1$. Nevertheless numerical experiments show an optimal convergence rate of 6.



k	Observed convergence rate
3	4.0191
5	6.0178
7	8.0096
9	9.9865

Figure 4: Approximation order of quasiinterpolation with normal perturbation family.

5 Implementation and Applications

Note that to evaluate the quasiinterpolation operator $Q^n_M f$ we need to compute $\mathfrak{t}^n_i = \mathcal{S}_n \mathbf{t}^n \in \ell\left([n], \mathbb{R}^D\right)$. We use Newtons Method with the system of equations

$$\mathbf{g}\left((\mathrm{circ}_\Lambda \mathfrak{f}^n_{\mathfrak{T}})_i\right) = \mathbf{g}\left((\mathrm{circ}_\Lambda \left(\mathfrak{f}^n + \mathbf{N}(\mathfrak{f}^n)\mathfrak{t}^n\right))_i\right) = 0$$

for all $i \in [n]$ to solve for $\mathfrak{t}^n$ numerically. Since

$$\frac{d\mathbf{g}\left((\mathrm{circ}_\Lambda \mathfrak{f}^n_{\mathfrak{T}})_i\right)}{d\mathfrak{t}^n_j} = 0$$

whenever $|i - j \mod n| > S$ the Matrix of the equation which we need to solve is sparse with nonzero entries on the diagonal band with bandwidth $(2S+1)\cdot D$, on the upper right triangle with side length $S \cdot D$ and on the lower left triangle with side length $S \cdot D$. Hence one Newtoniteration can be done in linear time with respect to n. As starting values we choose $\mathfrak{t}^n_i = 0$.

5.1 Approximation for Sphere-Valued Functions

In this case we can set $\mathbf{g}(x) = x^T x - 1$ and $\mathbf{N}(x) = x$ for all $x \in M$. With the substitution $\mathfrak{s}^n_i = \mathfrak{t}^n_i + 1$ we get the equations

$$\langle (\mathrm{circ}_\Lambda(\mathfrak{f}\mathfrak{s}))_i , (\mathrm{circ}_\Lambda(\mathfrak{f}\mathfrak{s}))_i \rangle - 1 = 0$$

for all $i \in [n]$ where $\mathfrak{f}\mathfrak{s}$ is defined by $(\mathfrak{f}\mathfrak{s})_i = \mathfrak{f}_i \mathfrak{s}_i$.
Measured approximation orders for the function $\mathbf{f}\colon \mathbb{T} \to S^3$ defined by

$$\mathbf{f}(t) = \begin{pmatrix} \sin(x(t))\sin(y(t))\sin(z(t)) \\ \sin(x(t))\sin(y(t))\cos(z(t)) \\ \sin(x(t))\cos(y(t)) \\ \cos(x(t)) \end{pmatrix}$$

where $x(t) = \sin(2t\pi)$, $y(t) = \cos(2t\pi)$ and $z(t) = \sin(2xt\pi)$ are displayed in Figure 4. We observe that quasiinterpolation with normal perturbation yields optimal convergence rate.

5.2 Approximation for Functions with Values in the Special Orthogonal Group

In this case we can set $\mathbf{g}(X) = X^T X - I_k$ where I_k is the $k \times k$ identity matrix. The space orthogonal to $SO(n)$ at a point $X \in SO(k)$ can be generated by left multiplication of X with a symmetric matrix. We therefore have to solve the equations

$$(\mathrm{circ}_\Lambda(\mathfrak{s}\mathfrak{f}))_i^T (\mathrm{circ}_\Lambda(\mathfrak{s}\mathfrak{f}))_i - I_k = 0$$

for all $i \in [n]$ where we seek the sequence of symmetric matrices $\mathfrak{s}$.

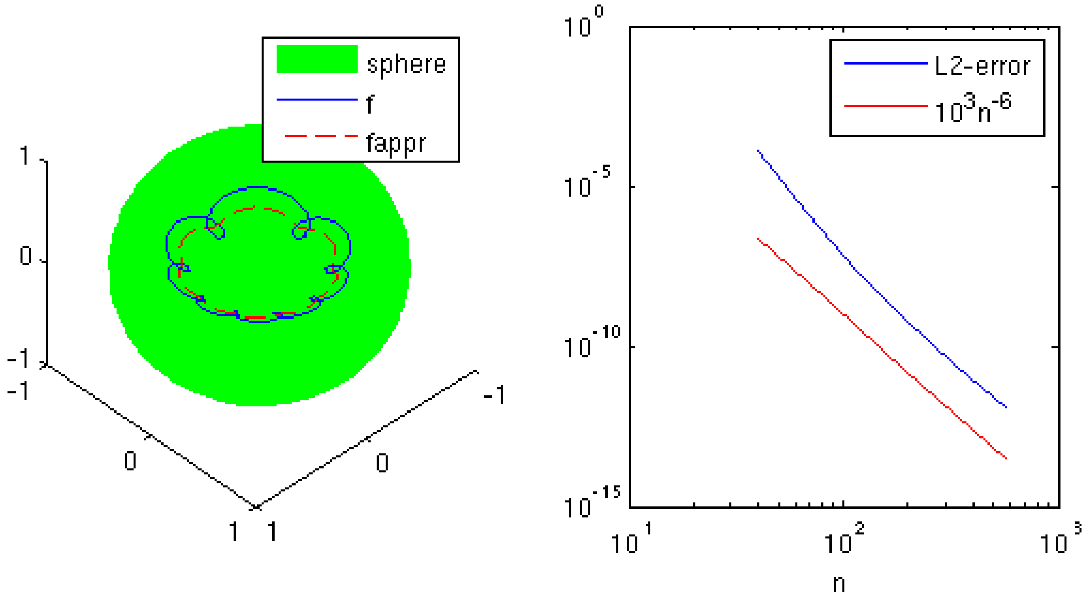


Figure 5: Fairing of a regular sphere-valued function

5.3 Fairing of a regular sphere-valued function

For a function $\mathbf{f} \in W^{1,2}(\mathbb{T}, M)$ we want to minimize the functional

$$J(\mathbf{g}) = \int_0^1 d(\mathbf{g}(t), \mathbf{f}(t))^2 + \lambda \|\nabla \mathbf{g}(t)\|^2 dt$$

where d is the distance function on M. Let $\mathbf{u}$ be the minimizer of J w.r.t. $W^{1,2}(\mathbb{T}, M)$ and for $n \in \mathbb{N}$ let $\mathbf{u}_n$ be the minimizer of J w.r.t. V_M^n. Using the results of [12] one can prove that

$$\|\mathbf{u}_n - \mathbf{u}\|_{L^2(\mathbb{T},M)} \lesssim \min_{\mathbf{v} \in V_M^n} \|\mathbf{v} - \mathbf{u}\|_{L^2(\mathbb{T},M)}.$$

By Lemma 3.5 we have

$$\min_{\mathbf{v} \in V_M^n} \|\mathbf{v} - \mathbf{u}\|_{L^2(\mathbb{T},M)} \lesssim n^{-(k+1)}.$$

Experiments were done for the function $\mathbf{f} \colon \mathbb{T} \to S^2$ defined by

$$\mathbf{f}(t) = \frac{\mathbf{a}(t)}{\|\mathbf{a}(t)\|_2} \qquad \text{with} \qquad \mathbf{a}(t) = \begin{pmatrix} 1 + \sin(2\pi t) + .2\sin(18\pi t) \\ 1 + \cos(2\pi t) + .2\cos(18\pi t) \\ 1 \end{pmatrix}$$

and $\lambda = 10^{-3}$. The functional is minimized using Newtons Method. A reference solution was computed with $n = 1000$. Result is shown in Figure 5.

5.4 Level Sets on the sphere

Assume that we are given a two times differentiable function $b : S^2 \to \mathbb{R}$ and $c \in \mathrm{Im}(g)$. We further assume that $b^{-1}(c)$ is homeomorphic to S^1 and $\nabla b(x) \neq 0$ for all $x \in b^{-1}(c)$. Let $\mathbf{r} \in b^{-1}(c)$ and $\mathbf{f} : \mathbb{T} \to S^2$ be the parametrization of $b^{-1}(c)$ with $\mathrm{Im}(\mathbf{f}) = b^{-1}(c)$, $\|d\mathbf{f}/dt\|_2 = \text{const}$, $\mathbf{f}(0) = \mathbf{r}$ and $\langle \nabla b(\mathbf{f}(t)) \times d\mathbf{f}(t)/dt, \mathbf{f}(t) \rangle > 0$ for all $t \in \mathbb{T}$. We will approximate $\mathbf{f}$ in two stages. In the first stage we define for $dl > 0$ small enough a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}_0}$ by $\mathbf{x}_0 = \mathbf{r}$, $\mathbf{x}_{i+1} \in S^2 \cap b^{-1}(c)$ and $\|\mathbf{x}_{i+1} - \mathbf{x}_i\|_2 = dl$. Numerically we can find the sequence by iteratively solving Newton Problems. Let $n^* \gg n$ be the smallest positive integer such that $\|\mathbf{x}_0 - \mathbf{x}_{n^*}\|_2 < dl$. Then for $n \in \mathbb{N}$ we have $\mathbf{f}_i^n \approx \mathbf{x}_{\lfloor n^* \cdot i/n \rfloor}$ for all $i \in [n]$. We

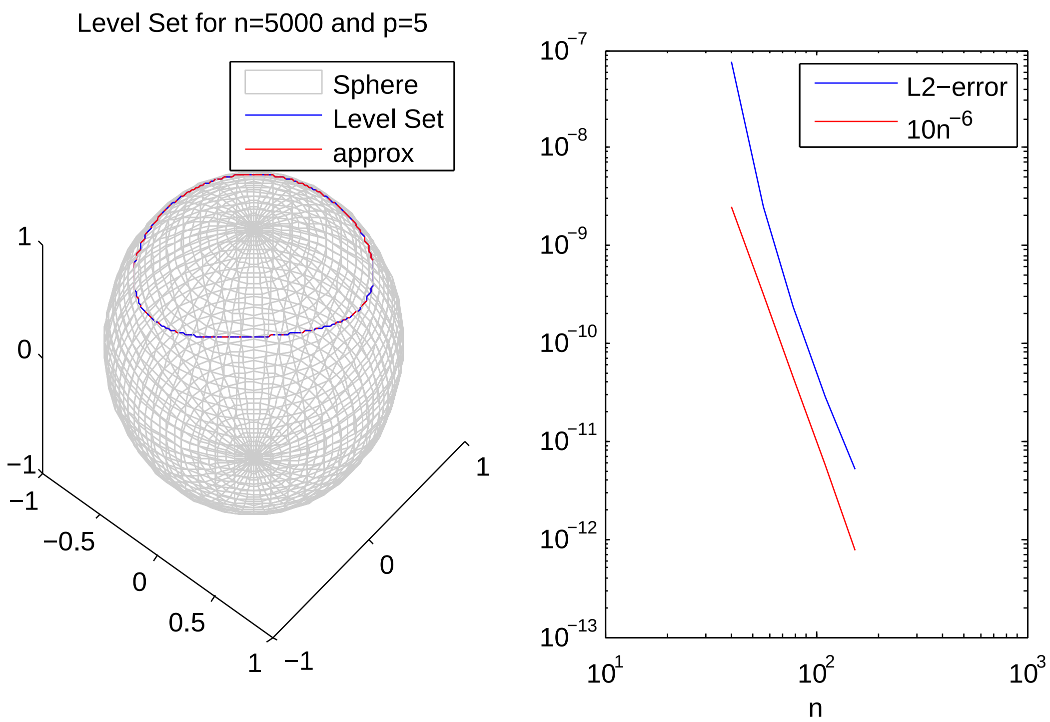


Figure 6: Level Sets on the Sphere

can apply the quasiinterpolation operator Q_M^n developed in this paper to the data $(\mathbf{x}_{\lfloor n^* \cdot i/n \rfloor})_{i=1}^n$ to get the sequence $y_i^n = Q_M^n(\mathbf{x}_{\lfloor n^* \cdot i/n \rfloor})_{i=1}^n \in \left(S^2\right)^n$ which is an approximation for $Q_M^n \mathbf{f}^n$.
In the second stage we minimize the functional $J : V_M^n \to \mathbb{R}$ given by

$$J(\mathbf{g}) = \int_0^1 (b(\mathbf{g}(t)) - c)^2 \, dt + \int_0^1 \left\| \frac{d\mathbf{g}(t)}{dt} \right\|_2^2 dt - \left(\int_0^1 \left\| \frac{d\mathbf{g}(t)}{dt} \right\|_2 dt \right)^2 + \|\mathbf{g}(0) - \mathbf{r}\|^2$$

using Newtons Method. We start with the function

$$t \mapsto P_M \left(\sum_{i \in [n]} y_i^n \Phi_n^{\mathrm{PER}}(t - i/n) \right)$$

which uses the values y_i^n from the first stage. As in 5.3 one can show that

$$\|\mathbf{u}_n - \mathbf{u}\|_{L^2(\mathbb{T},M)} \lesssim n^{-(k+1)}.$$

In Figure 6 numerical result for the function $b(x, y, z) = e^z + xyz$ and $c = 2$ are shown.

References

[1] M. Ainsworth, G. Andriamaro, and O. Davydov. Bernstein-bézier finite elements of arbitrary order and optimal assembly procedures. *SIAM Journal on Scientific Computing*, 33(6):3087–3109, 2011.

[2] D. Amsallem, J. Cortial, K. Carlberg, and C. Farhat. A method for interpolating on manifolds structural dynamics reduced-order models. *International journal for numerical methods in engineering*, 80(9):1241–1258, 2009.

[3] C. K. Chui. *An introduction to wavelets*. Academic Press Professional, Inc., 1992.

[4] P. Ciarlet. *The Finite Element Method for Elliptic Problems*. SIAM, 2002.

[5] P. Clément. Approximation by finite element function using local regularization. *RAIRO Anal. Numer.*, 2:77–84, 1975.

[6] J. A. Cottrell, T. J. Hughes, and Y. Bazilevs. *Isogeometric analysis: toward integration of CAD and FEA*. John Wiley & Sons, 2009.

[7] G. Deslauriers and S. Dubuc. Symmetric iterative interpolation processes. *Constructive approximation*, 5(1):49–68, 1989.

[8] N. Dyn, P. Grohs, and J. Wallner. Approximation order of interpolatory nonlinear subdivision schemes. *Journal of computational and applied mathematics*, 233(7):1697–1703, 2010.

[9] L. C. Evans. *Partial differential equations*. American Mathematical Society, 1998.

[10] P. Grohs. Approximation order from stability for nonlinear subdivision schemes. *Journal of Approximation Theory*, 162(5):1085–1094, 2010.

[11] P. Grohs. A general proximity analysis of nonlinear subdivision schemes. *SIAM Journal on Mathematical Analysis*, 42(2):729–750, 2010.

[12] P. Grohs, H. Hardering, and O. Sander. Optimal a priori discretization error bounds for geodesic finite elements. Technical report, ETH Zurich, 2013.

[13] P. Grohs and J. Wallner. Interpolatory wavelets for manifold-valued data. *Applied and Computational Harmonic Analysis*, 27(3):325–333, 2009.

[14] X. Pennec, P. Fillard, and N. Ayache. A riemannian framework for tensor computing. *International Journal of Computer Vision*, 66(1):41–66, 2006.

[15] I. U. Rahman, I. Drori, V. C. Stodden, D. L. Donoho, and P. Schröder. Multiscale representations for manifold-valued data. *Multiscale Modeling & Simulation*, 4(4):1201–1232, 2005.

[16] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3. de Gruyter, 1996.

[17] O. Sander. Geodesic finite elements on simplicial grids. *International Journal for Numerical Methods in Engineering*, 92(12):999–1025, 2012.

[18] O. Sander. Geodesic finite elements of higher order. 2013.

[19] G. O. Thorin. *Convexity theroems generalizing those of m. riesz and hadamard with some applications*. PhD thesis, Lund University, 1948.

[20] A. Weinmann. Interpolatory multiscale representation for functions between manifolds. *SIAM Journal on Mathematical Analysis*, 44(1):162–191, 2012.

[21] N. Wiener. Tauberian theorem. *Ann. Math.*, (33):1100, 1932.

[22] G. Xie and T. P.-Y. Yu. Approximation order equivalence properties of manifold-valued data subdivision schemes. *IMA Journal of Numerical Analysis*, 32(2):687–700, 2012.

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