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Research Report No. 2013-12  
April 2013

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# MULTI-LEVEL MONTE-CARLO FINITE ELEMENT METHODS FOR STOCHASTIC ELLIPTIC VARIATIONAL INEQUALITIES

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ABSTRACT. Multi-Level Monte-Carlo Finite Element (MLMC-FE) methods for the solution of stochastic elliptic variational inequalities are introduced, analyzed, and numerically investigated. Under suitable assumptions on the random diffusion coefficient, the random forcing function, and the deterministic obstacle, we prove existence and uniqueness of solutions of “mean-square” and “pathwise” formulations. Suitable regularity results for deterministic, elliptic obstacle problems lead to uniform pathwise error bounds, providing optimal-order error estimates of the statistical error and upper bounds for the corresponding computational cost for classical Monte-Carlo and novel MLMC-FE methods. Utilizing suitable multigrid solvers for the occurring sample problems, in two space dimensions MLMC-FE methods then provide numerical approximations of the expectation of the random solution with the same order of efficiency as for a corresponding deterministic problem, up to logarithmic terms. Our theoretical findings are illustrated by numerical experiments.

## 1. INTRODUCTION

Monte-Carlo (MC) methods are well established in statistical simulation. For partial differential equations (PDEs for short) with random coefficients, numerical realization of one MC “sample” entails the numerical solution of one deterministic PDE. Many of such “paths” are required for sufficient accuracy, causing suboptimal efficiency even if optimal algebraic solvers are used (see, e.g., [5, 6, 7, 19, 30]). Multi-level versions of MC were introduced, to the authors’ knowledge, by M. Giles [20, 21] for the numerical solution of Itô stochastic ordinary differential equations, following basic ideas in earlier work by S. Heinrich [27] on numerical quadrature. Such Multi-Level Monte-Carlo (MLMC) methods have been shown to provide similar efficiency for certain stochastic PDEs as in the corresponding deterministic case [8, 33, 34].

In the present paper, we consider elliptic obstacle problems with stochastic coefficients. Such problems arise, e.g., in the numerical simulation of subsurface flow problems or contact problems in mechanics with uncertain constitutive equations, specifically elastic moduli or friction coefficients (see, e.g., [35, 36, 38] and the references cited therein). Key characteristics of elliptic variational inequalities with stochastic coefficients are *low spatial regularity of the permeability*, *small spatial correlation lengths* (this implies slow convergence of Karhunen-Loève expansions), and the possible *nonstationarity* of realistic stochastic models, particularly from computational geosciences. All these factors hinder the efficient numerical simulation

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*Date:* April 7, 2013.

*Key words and phrases.* Multi-Level Monte-Carlo, Stochastic Partial Differential Equations, Stochastic Finite Element Methods, Multi-Level Methods, Variational Inequalities .

This research was supported in part under ERC AdG Grant STAHPDE No. 247277. The authors also want to thank Carsten Gräser (FU Berlin) for fruitful discussions. In particular, the numerical strategy for computing the number of samples  $M_l$  was suggested by him.

of such problems by spectral methods [18]. As for unconstrained problems, MC methods suffer from their typical lack of efficiency, even though fast multi-level solvers for discretized symmetric obstacle problems are available (see the review article [23] and the references cited therein). On this background, the present paper is devoted to the development, analysis and implementation of Multi-Level Monte-Carlo Finite Element Methods (MLMC-FEM for short) for symmetric, second order, elliptic obstacle problems with random coefficients.

In this paper, the notion of randomness of diffusion coefficients is based on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , with  $\Omega$  denoting the set of all elementary events. We consider isotropic random diffusion coefficients  $a(\cdot, \omega)$  as defined on an open, bounded Lipschitz polyhedron  $D \subset \mathbb{R}^d$ ,  $d = 1, 2, 3$ , for all  $\omega \in \Omega$ . These are strongly measurable mappings

$$(1.1) \quad \Omega \ni \omega \mapsto a(\cdot, \omega) \in L^\infty(D)$$

where we endow the space  $L^\infty(D)$  of realizations of diffusion coefficients with the sigma algebra of Borel sets to render it a measurable space. Though our algorithms will be well-defined even for random coefficients whose realizations are merely in  $L^\infty(D)$ , we will impose stronger, spatial continuity (either  $\mathbb{P}$ -a.s. Hölder continuity or  $\mathbb{P}$ -a.s. continuous differentiability in  $\overline{D}$ ) on the draws of the random coefficient in (1.1) in order to have convenient access to known regularity results for deterministic variational inequalities. In practice, however, often only a weaker “pathwise” regularity  $a(\cdot, \omega) \in C^{0,s}(\overline{D})$  holds for some  $0 < s < 1$ . Here  $C^{0,s}(\overline{D})$  denotes the (separable) Banach space of functions which are Hölder continuous with exponent  $0 < s < 1$  in  $\overline{D}$ . In the case  $s = 1$ , we consider the (separable) Banach space  $C^1(\overline{D})$  rather than the (nonseparable) Banach space  $C^{0,1}(\overline{D})$ . For a given random source  $f$ , i.e., a strongly measurable mapping

$$(1.2) \quad \Omega \ni \omega \mapsto f(\cdot, \omega) \in L^2(D),$$

and a deterministic obstacle function

$$(1.3) \quad \chi \in H^2(D), \quad \chi \leq 0 \text{ in } D,$$

we consider the stochastic obstacle problem which, formally in strong form, amounts to finding a random solution  $u(\cdot, \omega)$  such that for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  and for a.e.  $x \in D$  there holds

$$(1.4) \quad \begin{aligned} -\operatorname{div}(a(\cdot, \omega)\nabla u) &\geq f(\cdot, \omega) \text{ in } D, \\ u &\geq \chi \text{ in } D, \\ (\operatorname{div}(a(\cdot, \omega)\nabla u) + f(\cdot, \omega))(u - \chi) &= 0 \text{ in } D, \\ u|_{\partial D} &= 0. \end{aligned}$$

We concentrate on deterministic obstacle functions, because random obstacles  $\chi(\cdot, \omega)$  can be traced back to the deterministic obstacle zero by introducing the new variable  $w = u - \chi$ . A direct treatment of stochastic obstacles is contained in [10].

The solution  $u$  of the stochastic obstacle problem (1.4) not only depends on  $x \in D$ , but also on the “stochastic parameter”  $\omega \in \Omega$ . We prove existence and uniqueness of solutions  $u(\cdot, \omega)$  of a “pathwise” variational formulation of the deterministic obstacle problem (1.4) for  $\mathbb{P}$ -a.e. realization  $\omega \in \Omega$ . Taking expectations of the pathwise variational form, we arrive at a “mean-square” variational formulation. We show existence and uniqueness of a solution  $u$  with finite second moments. Regularity and uniform stability for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  of pathwise weak solutions follows from suitable regularity assumptions on the (random) data via regularity results for the deterministic, elliptic obstacle problem. From these regularity results, we obtain pathwise, optimal-order error estimates for continuous, piecewise linear Finite Element approximations. These error bounds hold uniformly for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . Our

arguments strongly rely on well-known Finite Element convergence results for deterministic problems (cf., e.g. [15, 37]).

The pathwise results are the basis for the efficient computation of the expectation value of the mean-square solution of the stochastic obstacle problem (1.4) by Monte-Carlo-type methods. We first prove that the classical Monte-Carlo (MC) method converges with the order  $1/2$  in terms of the number  $M$  of MC samples. Then, we show that, up to a logarithmic factor, the resulting Monte-Carlo Finite Element Method (MC-FEM) requires suboptimal computational cost of order  $N^3$ ,  $N^2$ ,  $N^{1+\frac{2}{3}}$  for space dimension  $d = 1, 2, 3$ , respectively, in terms of the number  $N$  of degrees of freedom used in the Finite Element approximation. Therefore, following Barth et al. [8], we introduce a Multi-Level-Monte-Carlo Finite Element Method (MLMC-FEM), which, in contrast to MC-FEM does not preserve conformity of the deterministic samples. Assuming that suitable algebraic solvers for the pathwise sample problems are available, we show that MLMC-FEM provides optimal-order approximations at computational cost of order  $N^2$  for  $d = 1$  and of optimal order  $N$  for problems in  $d = 2, 3$  space dimensions, up to logarithmic factors.

The discretized pathwise sample problems, i.e., discretized deterministic obstacle problems with spatially varying coefficients, can be solved iteratively up to discretization error accuracy by recent multigrid methods [3, 4, 23, 40]. Mesh-independent polylogarithmic convergence rates, as typically observed in numerical computations, have been recently justified theoretically for so-called Standard Monotone Multi-Grid Methods (STDMMG) [29, 31] by Badea [4] in  $d = 1, 2$  space dimensions. Hence, for  $d = 2$  space dimensions MLMC-FEM with algebraic STDMMG solver turns out to provide an optimal-order approximation of the expectation of the random solution  $u$  at a computational cost which is essentially the same as for a corresponding deterministic problem. These theoretical results are illustrated by numerical experiments in one and two space dimensions using model problems with known solutions.

The paper is organized as follows. In the following Section 2, we collect basic properties of random fields and *Elliptic Variational Inequalities* (EVIs) which shall be used in the ensuing developments. In Section 3, we state the assumptions on the random diffusion coefficient  $a(\cdot, \omega)$ , the random source term  $f(\cdot, \omega)$ , the deterministic obstacle  $\chi$ , and the spatial domain  $D$ , and discuss possible generalizations along with typical examples. We also provide pathwise and mean-square formulations of the stochastic obstacle problem (1.4) and present results on existence, uniqueness, measurability, summability, regularity, and stability of the random solution. Section 4 first addresses the convergence analysis of a stochastic Finite Element approximation of the pathwise variational formulation of (1.4) together with the analysis of convergence and computational cost of MC-FEM and MLMC-FEM and then algebraic multigrid solution of the pathwise sample problems. As a corollary, we obtain almost optimal efficiency of MLMC-FEM with algebraic STDMMG solver for  $d = 2$  space dimensions, which is one of the main results of this paper. The concluding Section 5 contains several numerical experiments illustrating our theoretical findings.

## 2. PRELIMINARIES

**2.1. Random Fields.** Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space, where  $\Omega$  denotes a set of elementary events,  $\mathcal{A} \subset 2^\Omega$  stands for the  $\sigma$ -algebra of all possible events, and  $\mathbb{P} : \mathcal{A} \rightarrow [0, 1]$  is a probability measure. Then, for any separable Banach space  $X$  of real-valued functions on the domain  $D \subset \mathbb{R}^d$  with norm  $\|\cdot\|_X$ , we introduce the Bochner space of strongly measurable,

$r$ -summable mappings  $v : \Omega \rightarrow X$  by (see, e.g., [16, Chap.1])

$$L^p(\Omega, \mathcal{A}, \mathbb{P}; X) := \{v : \Omega \rightarrow X \mid v \text{ strongly measurable, } \|v\|_{L^p(\Omega; X)} < \infty\},$$

where, for  $0 < p \leq \infty$ ,

$$\|v\|_{L^p(\Omega; X)} := \begin{cases} \left( \int_{\Omega} \|v(\cdot, \omega)\|_X^p d\mathbb{P}(\omega) \right)^{1/p} & \text{if } 0 < p < \infty, \\ \text{ess sup}_{\omega \in \Omega} \|v(\cdot, \omega)\|_X & \text{if } p = \infty. \end{cases}$$

In the following, we shall not explicitly indicate the dependence of Bochner spaces and their norms on the probability measure  $\mathbb{P}$ , if this measure is clear from the context.

Let  $\mathcal{B} \in \mathcal{L}(X, Y)$  denote a continuous linear mapping from  $X$  to a separable Hilbert space  $Y$  with norm  $\|\mathcal{B}\|_{X, Y}$ . For a random field  $v \in L^p(\Omega; X)$  this mapping defines a random field  $\mathcal{B}v \in L^p(\Omega; Y)$  with the property

$$\|\mathcal{B}v\|_{L^p(\Omega; Y)} \leq \|\mathcal{B}\|_{X, Y} \|v\|_{L^p(\Omega; X)}.$$

Furthermore, there holds

$$\mathcal{B} \int_{\Omega} v d\mathbb{P}(\omega) = \int_{\Omega} \mathcal{B}v d\mathbb{P}(\omega).$$

We refer to Chapter 1 of [16] for further results on Banach space valued random variables.

**2.2. Elliptic Variational Inequalities (EVIs).** We briefly recall some basic existence results on deterministic EVIs, in particular the theorem of Stampaccia [28]. Let  $\mathcal{V}$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ , associated norm  $\|\cdot\|_{\mathcal{V}}$  and dual space  $\mathcal{V}^*$ . We recall that a bilinear form  $b(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$  is

i) *continuous*, if there exists  $C_1 > 0$  such that

$$(2.1) \quad |b(u, v)| \leq C_1 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} \quad \forall u, v \in \mathcal{V},$$

ii) *coercive*, if there exists  $C_2 > 0$  such that

$$(2.2) \quad b(u, u) \geq C_2 \|u\|_{\mathcal{V}}^2 \quad \forall u \in \mathcal{V}.$$

**Theorem 2.1.** *Let  $b(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$  be a continuous, coercive, not necessarily symmetric bilinear form on the Hilbert space  $\mathcal{V}$ , and let  $\emptyset \neq \mathcal{K} \subset \mathcal{V}$  be a closed, convex subset. Then, for any  $\ell \in \mathcal{V}^*$  there exists a unique solution  $u \in \mathcal{K}$  of the EVI*

$$(2.3) \quad u \in \mathcal{K} : \quad b(u, v - u) \geq \ell(v - u) \quad \forall v \in \mathcal{K}.$$

In the iterative solution of discretized deterministic obstacle problems as described in Section 4.5 later on, we use the following reformulation of (2.3) in terms of convex minimization that exclusively holds in the symmetric case.

**Proposition 2.2.** *If the bilinear form  $b(\cdot, \cdot)$  is symmetric, then the unique solution of the EVI (2.3) is characterized as the unique minimizer of the associated potential*

$$(2.4) \quad J(v) := \frac{1}{2} b(v, v) - \ell(v), \quad v \in \mathcal{V},$$

over the closed, convex cone  $\mathcal{K}$ , i.e.,

$$(2.5) \quad u = \arg \min \{J(v) : v \in \mathcal{K}\}.$$

For given  $v \in \mathcal{V}$  the error  $\|u - v\|_b$  in the energy norm  $\|\cdot\|_b = b(\cdot, \cdot)^{1/2}$  and the energy error are related according to

$$(2.6) \quad \frac{1}{2}\|u - v\|_b^2 \leq J(v) - J(u) \leq \|u - v\|_b \left( \frac{1}{2}\|u - v\|_b + \|u - u^*\|_b \right).$$

Here  $u^*$  stands for the unconstrained solution, i.e.,  $b(u^*, v) = \ell(v)$ . Note the mismatch between the lower and upper bound which does not occur in the unconstrained case  $\mathcal{K} = \mathcal{V}$ .

We shall also be interested in the following special case.

**Proposition 2.3.** *Assume that*

$$(2.7) \quad \mathcal{K} \subset \mathcal{V} \quad \text{is a closed, convex cone with vertex } 0.$$

*Then the solution  $u \in \mathcal{K}$  of the EVI (2.3) is characterized by*

$$(2.8) \quad u \in \mathcal{K} : \quad b(u, v) \geq \ell(v) \quad \forall v \in \mathcal{K} \quad \text{and} \quad b(u, u) = \ell(u).$$

*Moreover, with the constant  $C_2$  as in (2.2) there holds the a-priori estimate*

$$(2.9) \quad \|u\|_{\mathcal{V}} \leq \frac{1}{C_2} \|\ell\|_{\mathcal{V}^*}.$$

*Proof.* Let  $u \in \mathcal{K}$  be a solution of (2.3). As  $\mathcal{K}$  is closed under linear combinations with positive coefficients, we have  $w = u + v \in \mathcal{K}$  for all  $v \in \mathcal{K}$ . Inserting  $w = u + v$  into (2.3) implies  $b(u, w) \geq \ell(w)$  for all  $w \in \mathcal{K}$ . Inserting  $w = u$  into this inequality and  $v = 0 \in \mathcal{K}$  into (2.3), we get  $b(u, u) = \ell(u)$ . Conversely, if  $u$  solves (2.8), then we can subtract the equality from the inequality in (2.8) to show that  $u$  solves (2.3). The estimate (2.9) is a straightforward consequence of the reformulation (2.8).  $\square$

### 3. ELLIPTIC OBSTACLE PROBLEM WITH STOCHASTIC COEFFICIENTS

After these preparations, we now turn to the variational formulation of the unilateral stochastic boundary value problem (1.4). To this end, we first introduce a ‘‘pathwise’’ abstract formulation which closely resembles the deterministic formulation (2.3) and verify its well-posedness. We then present examples of the abstract problem which, in particular, are not uniformly elliptic.

**3.1. Random Diffusion Coefficients.** We assume that the stochastic diffusion coefficient  $a(x, \omega)$  is, possibly after modification on a null-set, well-defined and computationally accessible for every  $\omega \in \Omega$ . To ensure well-posedness later on, we impose the following assumptions on the random coefficient  $a(x, \omega)$ , the random source term  $f$  and the deterministic obstacle function  $\chi$ .

**Assumption 3.1.** The random diffusion coefficient  $a(\cdot, \omega)$  and the right hand side  $f(\cdot, \omega)$ , are strongly measurable mappings  $\Omega \ni \omega \mapsto a(\cdot, \omega) \in L^\infty(D)$  and  $\Omega \ni \omega \mapsto f(\cdot, \omega) \in L^2(D)$ , respectively. The random diffusion coefficient  $a(\cdot, \omega)$  is elliptic in the sense that there exist real-valued random variables  $\check{a}, \hat{a}$  such that

$$(3.1) \quad 0 < \check{a}(\omega) \leq a(x, \omega) \leq \hat{a}(\omega) < \infty \quad \text{a.e. } x \in D$$

holds for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . We have  $f(\cdot, \omega) \in L^2(D)$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , and the obstacle function  $\chi \in H^2(D)$  satisfies  $\chi \leq 0$  in  $D$ .

In deriving optimal convergence rates for Finite Element discretizations (cf. Proposition 3.7 ahead) we will sharpen Assumption 3.1 by the following uniform ellipticity and regularity of the random coefficient  $a(\cdot, \omega)$ .

**Assumption 3.2.** a) (uniform ellipticity) There exist constants  $a_-$ ,  $a_+$  such that the random variables  $\check{a}$  and  $\hat{a}$  in (3.1) satisfy  $\mathbb{P}$ -almost surely

$$(3.2) \quad 0 < a_- \leq \check{a}(\omega) \leq \hat{a}(\omega) \leq a_+ < \infty ,$$

and the right hand side satisfies  $f \in L^2(\Omega; L^2(D))$ .

b) (almost sure spatial regularity of  $a$ ) The random diffusion coefficient  $a(\cdot, \omega)$  is “pathwise” Lipschitz continuous in the sense that  $a$  is a measurable map

$$(3.3) \quad \Omega \ni \omega \mapsto a(\cdot, \omega) \in C^1(\overline{D})$$

with the property

$$(3.4) \quad a \in L^\infty(\Omega; C^1(\overline{D})) .$$

c) (regularity of  $D$ ) The spatial domain  $D \subset \mathbb{R}^d$  is convex.

The above Assumption 3.1 is satisfied, for example, for lognormal Gaussian random fields  $a$  with the choice

$$(3.5) \quad 0 < \check{a}(\omega) := \operatorname{ess\,inf}_{x \in D} a(x, \omega) , \quad \hat{a}(\omega) := \operatorname{ess\,sup}_{x \in D} a(x, \omega) < \infty .$$

We refer to [14, Proposition 2.3], which in this case states

$$(3.6) \quad \hat{a} \in L^p(\Omega) , \quad (\check{a})^{-1} \in L^p(\Omega) \quad \text{for all } 0 < p < \infty .$$

For lognormal Gaussian random fields  $a$  with sufficiently smooth covariance kernel function  $R_a(\cdot, \cdot)$ , the decay of the Karhúnen-Loève eigenvalues to zero increases with smoothness (see, e.g., [39, Appendix]). Then, the  $C^1(\overline{D})$  regularity in sense of (3.3) in Assumption 3.2 b) is satisfied in mean square, but not  $\mathbb{P}$ -a.s.. Hence, we will assume occasionally that

$$(3.7) \quad a \in L^p(\Omega; C^1(\overline{D})) \quad \text{for all } 0 < p < \infty .$$

and that the mean diffusion coefficient  $\bar{a}$  satisfies

$$(3.8) \quad \bar{a} = \mathbb{E}[a] \in C^1(\overline{D}) .$$

We emphasize that the extra Assumption 3.2 or (3.7) and (3.8) are imposed only to ensure  $\mathbb{P}$ -a.s. sufficient regularity of the random solution to yield full first order convergence of continuous, piecewise linear Finite Element discretizations.

Moreover, assumption (3.3) also implies that the covariance kernel  $R_a$  of  $a$ , defined by

$$R_a := \mathbb{E}[(a - \mathbb{E}[a]) \otimes (a - \mathbb{E}[a])] \in C^1(\overline{D} \times \overline{D})$$

induces a self-adjoint integral operator  $\mathcal{C}_a$ , the *covariance operator*, which is compact from  $L^2(D)$  to  $L^2(D)$ , via

$$(\mathcal{C}_a \varphi)(x) = \int_{x' \in D} R_a(x, x') \varphi(x') dx' , \quad x \in D .$$

The spectral theorem for compact, self-adjoint operators implies that  $\mathcal{C}_a$  has a countable sequence  $(\lambda_k, \varphi_k)_{k \geq 1}$  of eigenpairs with the sequence  $\{\lambda_k\}_{k \geq 1}$  accumulating only at zero, and with a sequence of eigenfunctions  $\varphi_k \in L^2(D)$  which we assume to be an  $L^2(D)$ -orthogonal, dense set in  $L^2(D)$ , i.e., we assume that the covariance operator  $\mathcal{C}_a$  does not have finite rank or, equivalently, that the “noise” input  $a$  is genuinely infinite-dimensional. Next, we present several concrete examples of random diffusion coefficients  $a$  given in terms of their Karhúnen-Loève expansions.

**Example 3.1.** (*Uniform Random Field*)

Here, the random field  $a \in L^2(\Omega; C^1(\bar{D}))$  is assumed to satisfy the uniform ellipticity condition (3.2), i.e., we control the random coefficient with *deterministic* lower and upper bounds  $a_-$ ,  $a_+$ . The assumption  $a \in L^2(\Omega; C^1(\bar{D}))$  implies the pathwise spatial regularity  $a(\cdot, \omega) \in C^1(\bar{D})$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , and also that  $\varphi_k \in C^1(\bar{D})$ . We may expand the field  $a(\cdot, \omega) \in C^1(\bar{D})$  in a Karh unen-Lo eve series, i.e.,

$$a(x, \omega) = \bar{a}(x) + \sum_{k \geq 1} \sqrt{\lambda_k} Y_k(\omega) \varphi_k(x),$$

where, *assuming that the  $\varphi_k$  are normalized in  $L^2(D)$* , the random coefficients  $Y_k(\omega)$  are

$$Y_k(\omega) = \frac{1}{\sqrt{\lambda_k}} \int_{x' \in D} (a - \mathbb{E}[a])(x', \omega) \varphi_k(x') dx', \quad k = 1, 2, \dots$$

**Example 3.2.** (*Lognormal Gaussian Random Fields in  $D$* )

We assume the random field  $a$  to be such that for some deterministic  $\tilde{a} \in C^1(\bar{D})$  the field  $g = \log(a - \tilde{a})$  is a homogeneous, Gaussian random field in  $D$  with mean  $\bar{g} = \log(\bar{a}) \in C^1(\bar{D})$ , and with Lipschitz continuous covariance kernel

$$(3.9) \quad R_g(x, x') := \mathbb{E} [(g(x, \cdot) - \mathbb{E}[g](x))(g(x', \cdot) - \mathbb{E}[g](x'))] = \rho(\|x - x'\|), \quad x, x' \in D.$$

In (3.9), the function  $\rho(\cdot)$  is at least Lipschitz continuous. It is well known (see, e.g., [1]), that prescribing the (deterministic) functions  $\tilde{a}$ ,  $\bar{g}$  and  $\rho$ , the stationary, Gaussian random field  $g$  is determined, up to null-events.

Moreover, assuming only Lipschitz regularity of  $\rho(\cdot)$  near zero, the sample paths  $a(\cdot, \omega)$  belong  $\mathbb{P}$ -a.s. to  $C^{0,s}(\bar{D})$  with  $s < 1/2$  (see, e.g., [14, Proposition 2.1]). This is the case, e.g., for the so-called *exponential covariance function*. Here,  $\rho$  in (3.9) is given by  $\rho_{1/2}(r) = \sigma^2 \exp(-r/\lambda)$  where,  $\sigma > 0$  is the variance, and  $\lambda > 0$  is a correlation length parameter.

Additional smoothness of  $\rho$  near  $r = 0$  implies higher spatial regularity of the realizations  $a(\cdot, \omega)$ . For example, for the Gaussian covariance kernel, where  $\rho$  equals  $\rho_\infty(r) = \sigma^2 \exp(-r^2/\lambda^2)$  sample paths are infinite differentiable, in quadratic mean, in  $\bar{D}$  (see, e.g. [1, Chapter 8]).

Both kernel functions,  $\rho_{1/2}$  and  $\rho_\infty$ , the exponential and Gaussian covariance kernel, are special cases of the so-called *Matern-Covariances* (see, e.g., [32]) where  $\rho$  in (3.9) is given by

$$(3.10) \quad \rho_\nu(r) := \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left( 2\sqrt{\nu} \frac{r}{\lambda} \right)^\nu K_\nu(2\sqrt{\nu} \frac{r}{\lambda}).$$

Here,  $K_\nu$  denotes the modified Bessel function of the second kind. The smoothness of  $\rho_\nu(\cdot)$  at  $r = 0$  and, correspondingly, the spatial regularity of realizations of  $a(\cdot, \omega)$  depends on the parameter  $\nu$ .

**3.2. Pathwise Formulation and Well-Posedness.** For given probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and a separable Hilbert space  $V$ , we consider a stochastic analog of the abstract EVI (2.3). To this end, we assume that a given random bilinear form  $b(\omega; \cdot, \cdot) : V \times V \mapsto \mathbb{R}$  satisfies

$$(3.11) \quad |b(\omega; w, v)| \leq C_1(\omega) \|w\|_V \|v\|_V \quad \forall v, w \in V$$

and

$$(3.12) \quad b(\omega; v, v) \geq C_2(\omega) \|v\|_V^2 \quad \forall v \in V$$

for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  with random variables  $C_1(\omega), C_2(\omega) \in L^p(\Omega)$ ,  $2 \leq p < \infty$ , with the property

$$(3.13) \quad 0 < C_1(\omega) \leq C_2(\omega) < \infty, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega .$$

In later applications, these conditions will be assured by Assumption 3.1, in particular by pathwise ellipticity (3.1), or by the stronger uniform ellipticity (3.2).

We further assume given a closed, convex subset  $K \subset V$ . Then, for a linear functional  $\ell(\cdot) \in V^*$ , and for a given realization  $\omega \in \Omega$ , we consider the “pathwise” random EVI

$$(3.14) \quad u(\omega) \in K : \quad b(\omega; u(\omega), v - u(\omega)) \geq \ell(v - u(\omega)) \quad \forall v \in K .$$

**Theorem 3.3.** *Let the assumptions (3.11), (3.12), and (3.13) hold. Then the stochastic EVI (3.14) admits, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , a unique solution  $u(\omega) \in K$ . The solution map*

$$(3.15) \quad \Omega \ni \omega \mapsto \Sigma(\omega) := \{u \in K : u \text{ solves (3.14)}\}$$

*is measurable with respect to the Borel  $\sigma$ -algebra  $\mathcal{B}(V)$  of  $V$ , i.e.,  $u \in L^0(\Omega; V)$ .*

*Proof.* For each fixed realization  $\omega \in \Omega$  the random EVI (3.14) becomes a special case of the deterministic EVI (2.3). Hence, under the assumptions (3.11), (3.12), and (3.13), existence and uniqueness of a solution  $u(\omega) \in K$  of (3.14) follows from Theorem 2.1. The measurability of the solution correspondence (3.15) is shown, for example, in [25, Section 1].  $\square$

We note in passing that Theorem 3.3 does not require the bilinear form  $b(\omega; \cdot, \cdot)$  to be symmetric.

**3.3. Stochastic Variational Formulation and Well-Posedness.** Variational formulations of (1.4) will be based on the Hilbert space  $V = H_0^1(D)$  which is a closed, linear subspace of  $H^1(D)$ . By Poincaré’s inequality, the expressions

$$V \ni v \rightarrow \|v\|_V := \left( \int_D |\nabla v|^2 dx \right)^{1/2}, \quad \|v\|_a := \left( \int_D \bar{a} |\nabla v|^2 dx \right)^{1/2}$$

are equivalent norms on  $V$ . Throughout the following, we identify  $L^2(D)$  with its dual and denote by  $V^*$  the dual of  $V$  with respect to the “pivot” space  $L^2(D)$ , i.e., we work in the triplet  $V \subset L^2(D) \simeq L^2(D)^* \subset V^*$ . To state the variational formulation of the stochastic elliptic boundary value problem (1.4), we define the set

$$(3.16) \quad K := \{v \in H_0^1(D) : v \geq \chi \text{ a.e. } x \in D\}$$

with given, deterministic obstacle function  $\chi$  satisfying Assumption 3.1. Then  $K$  is a closed, convex subset of  $V$  and  $0 \in K$ , so that (2.7) is valid. For each realization  $\omega \in \Omega$ , the pathwise variational form of (1.4) is then given by (3.14) with

$$(3.17) \quad b(\omega; v, w) := \int_D a(x, \omega) \nabla v \cdot \nabla w \, dx, \quad \ell(\omega; w) := \int_D f(x, \omega) w \, dx, \quad v, w \in V,$$

and  $K$  defined in (3.16). The pathwise formulation (3.14) will be the basis of MC sampling.

Existence, uniqueness, and stability follow from Theorem 3.3 and Proposition 2.3.

**Proposition 3.4.** *Let Assumption 3.1 hold. Then the pathwise obstacle problem (3.14) has a unique solution  $u(\omega) \in K$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  which fulfills the a-priori estimate*

$$(3.18) \quad \|u(\cdot, \omega)\|_V \leq \frac{1}{\bar{a}(\omega)} \|f(\cdot, \omega)\|_{L^2(D)} \quad \mathbb{P}\text{-a.e. } \omega \in \Omega .$$

Imposing the stronger uniform ellipticity condition (3.2) we get the uniform estimate

$$\|u(\cdot, \omega)\|_V \leq \frac{1}{a_-} \|f(\cdot, \omega)\|_{L^2(D)} \quad \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

providing the moment bound  $u \in L^2(\Omega; V)$  for  $f \in L^2(\Omega; L^2(D))$ .

**Remark 3.5.** From the pathwise bound (3.18), moment bounds can be obtained *without condition* (3.2). Indeed, raising (3.18) to the power  $1 \leq r < \infty$ , and using Hölder's inequality with conjugate indices  $t, t' \geq 1$ ,  $1/t + 1/t' = 1$  gives

$$(3.19) \quad \|u\|_{L^r(\Omega; V)} \leq \|(\check{a})^{-1}\|_{L^{rt}(\Omega)} \|f\|_{L^{rt'}(\Omega; L^2(D))}.$$

Hence, imposing the condition (3.6), we can select  $p = rt < \infty$  sufficiently large, to find  $\|u\|_{L^r(\Omega; V)} < \infty$  provided that  $f \in L^{r(1+\delta)}(\Omega; L^2(D))$  for some  $\delta > 0$ .

As a second approach to the existence of solutions to the random EVI (1.4), we formally take expectations on both sides of this expression, and arrive at the “*mean-square*” *weak formulation* of the stochastic elliptic boundary value problem (1.4): given a random coefficient  $a(x, \omega)$ , a random source term  $f(\cdot, \omega)$ , and a deterministic lower obstacle  $\chi$  satisfying Assumption 3.1, we select

$$\mathcal{V} := L^2(\Omega; V), \quad \mathcal{K} := \overline{L^\infty(\Omega, \mathbb{P}; K)}^{\|\cdot\|_{\mathcal{V}}}$$

with the convex set  $K \subset V$  defined in (3.16). Then,  $\mathcal{K} \subset L^2(\Omega; V)$  is convex and closed, and  $\mathcal{V}^* \simeq L^2(\Omega; V^*)$ . The mean-square weak formulation of the random EVI (1.4) now reads

$$(3.20) \quad u \in \mathcal{K} : \quad B(u, v - u) \geq F(v - u) \quad \forall v \in \mathcal{K}.$$

Here the bilinear form  $B(\cdot, \cdot) : L^2(\Omega; V) \times L^2(\Omega; V) \rightarrow \mathbb{R}$  is defined by

$$B(v, w) = \mathbb{E} \left[ \int_D a(x, \omega) \nabla v(x, \cdot) \cdot \nabla w(x, \cdot) dx \right],$$

and  $F : L^2(\Omega; V) \mapsto \mathbb{R}$  takes the form

$$F(v) = \mathbb{E} [\ell(\cdot; v)] = \mathbb{E} \left[ \int_D f(x, \omega) v(x, \cdot) dx \right].$$

To show well-posedness of this integrated form of the sEVI, we additionally require  $f \in L^2(\Omega; D)$  and the uniform ellipticity condition (3.2) for the stochastic coefficient.

**Theorem 3.6.** *Let Assumptions 3.1 and 3.2 a) hold. Then the weak formulation (3.20) of the stochastic EVI (1.4) admits a unique solution  $u \in \mathcal{K}$ .*

*Proof.* Exploiting Assumption 3.1 and uniform ellipticity (3.2), we get the upper bound

$$(3.21) \quad |B(v, w)| \leq \text{ess sup}_{\omega \in \Omega} \|a(\cdot, \omega)\|_{L^\infty(D)} \|v\|_{L^2(\Omega; V)} \|w\|_{L^2(\Omega; V)} \leq a_+ \|v\|_{L^2(\Omega; V)} \|w\|_{L^2(\Omega; V)}$$

and, using again the uniform ellipticity (3.2), also the lower bound

$$(3.22) \quad B(v, v) \geq a_- \|v\|_{L^2(\Omega; V)}^2.$$

Moreover, for given  $f \in L^2(\Omega; V^*)$  we have for every  $w \in \mathcal{V}$

$$(3.23) \quad |F(w)| \leq \|f\|_{L^2(\Omega; L^2(D))} \|w\|_{L^2(\Omega; V)}.$$

The assertion now follows from Theorem 2.1. □

**3.4. Regularity of Solutions.** As a consequence of the regularity theory for *deterministic obstacle problems for the Laplacian* (see, e.g., [37, Chapter 5]), Assumptions 3.1 and 3.2 ensure that  $u(\cdot, \omega) \in H^2(D)$  holds,  $\mathbb{P}$ -a.s..

**Proposition 3.7.** *Let Assumptions 3.1 and 3.2 hold. Then the mean-square random obstacle problem (3.20) admits a unique solution  $u \in L^2(\Omega; W)$ , where the linear space  $W$  is defined by  $W := \{w \in V : \Delta w \in L^2(D)\}$  and is equipped with the norm*

$$\|w\|_W := \|\Delta w\|_{L^2(D)} + \|w\|_{L^2(D)} .$$

Further, there holds the a-priori estimate

$$(3.24) \quad \|u\|_{L^2(\Omega; W)} \leq C(a) \|f\|_{L^2(\Omega; L^2(D))} ,$$

where  $C(a)$  depends on  $a_-$  and  $a_+$  in (3.2).

*Proof.* As a consequence of Assumption 3.2, i.e., the  $\mathbb{P}$ -a.s.  $W^{1,\infty}(D)$ -regularity of the realizations of the stochastic coefficient  $a(\omega)$ , the realization  $u(\omega) = u(\cdot, \omega)$  of the random solution  $u \in L^2(\Omega; V)$  of the mean-square weak formulation (3.20) solves the deterministic obstacle problem for the Laplacian

$$(3.25) \quad u(\omega) \in K : \quad \int_D \nabla u(\omega) \cdot \nabla(v - u(\omega)) \, dx \geq \int_D \tilde{f}(v - u(\omega)) \, dx \quad \forall v \in K$$

for  $\mathbb{P}$ -a.e.  $\omega$  with the random source

$$\tilde{f}(\cdot, \omega) = \frac{1}{a(\cdot, \omega)} (f(\cdot, \omega) + \nabla a(\cdot, \omega) \cdot \nabla u(\cdot, \omega)) .$$

By Assumption 3.1, we have  $f(\cdot, \omega) \in L^2(D)$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  and Assumption 3.2 b) yields uniform regularity  $a(\cdot, \omega) \in C^1(\overline{D})$ . Together with the a-priori estimate (3.18) and uniform ellipticity  $a(\cdot, \omega) \geq \check{a}(\omega) \geq a_- > 0$ , as stated in Assumption 3.2 a), this implies

$$(3.26) \quad \|\tilde{f}(\cdot, \omega)\|_{L^2(D)} \leq C \|f(\cdot, \omega)\|_{L^2(D)} , \quad \mathbb{P}\text{-a.e. in } \Omega ,$$

with a constant  $C$  independent of  $\omega \in \Omega$ . As a consequence, we may estimate

$$(3.27) \quad \|\Delta u(\cdot, \omega)\|_{L^2(D)} \leq C \|f(\cdot, \omega)\|_{L^2(D)} , \quad \mathbb{P}\text{-a.e. in } \Omega ,$$

with a possibly different constant  $C$  independent of  $\omega \in \Omega$  by utilizing convexity of  $D$  (cf. Assumption 3.2 c)) together with well-known  $H^2(D)$  regularity results for the deterministic obstacle problem (3.25) (see, e.g., [2] or [37, Corollary 5:2.3]).

Adding the corresponding bound for  $\|u(\cdot, \omega)\|_{L^2(D)}$  (which follows from (3.18) and the Poincaré-inequality), raising both sides of the resulting bound on the  $\|\cdot\|_W$  norm of  $u$  to the power 2 and taking expectations implies the assertion (3.24).  $\square$

We recall that the first part (3.3) of Assumption 3.2 b) is satisfied, for example, for log-normal Gaussian random fields  $a$  with sufficiently regular covariance function  $R_a(x, x')$ . Note that we impose the (strong) regularity Assumption 3.2 b) only to attain full spatial regularity of solutions to the pathwise random EVIs (3.14) which in turn will provide (first order) convergence of continuous, piecewise linear Finite Element discretizations later on.

**Remark 3.8.** Under mere Lipschitz continuity of  $R_a(\cdot, \cdot)$ , Assumption 3.2 does not hold, in general. In this case, only the weaker statement  $\Omega \ni \omega \mapsto a(\cdot, \omega) \in C^{0,s}(\overline{D})$  with  $0 < s < 1/2$  is available  $\mathbb{P}$ -a.s. (see [14, Proposition 1]). However, the a-priori error estimates for MC-FE and MLMC-FE to be stated in Theorem 4.5 and Theorem 4.9, respectively, will remain valid in this case, albeit with lower convergence rates. Note that a lack of full regularity would

not affect the convergence analysis of multigrid methods for discretized pathwise EVIs (cf. Remark 4.12).

**Remark 3.9.** If uniform ellipticity (3.2) were dropped, and only Assumption 3.1 and pathwise regularity (3.3) is required, then the reasoning in the proof of Proposition 3.7 above is still valid, except for the last step: instead of (3.27), the representation (3.26) and well-known  $H^2(D)$ -regularity results for the deterministic obstacle problem for the Laplacian provide the  $\mathbb{P}$ -a.s. bound

$$\|\Delta u(\cdot, \omega)\|_{L^2(D)} \leq \frac{C}{\check{a}(\omega)} \left( \|f(\cdot, \omega)\|_{L^2(D)} + \|\nabla a(\cdot, \omega)\|_{L^\infty(D)} \|\nabla u(\cdot, \omega)\|_{L^2(D)} \right) .$$

Using the pathwise a-priori estimate (3.18) we arrive at the  $\mathbb{P}$ -a.s. bound

$$(3.28) \quad \|\Delta u(\cdot, \omega)\|_{L^2(D)} \leq \frac{C}{\check{a}(\omega)} \left( \|f(\cdot, \omega)\|_{L^2(D)} + \frac{\|\nabla a(\cdot, \omega)\|_{L^\infty(D)}}{\check{a}(\omega)} \|f(\cdot, \omega)\|_{V^*} \right) .$$

Note that (3.28) provides convergence of piecewise linear Finite Element approximations with optimal order. Squaring (3.28) and proceeding as in Remark 3.5, we obtain the desired  $L^2$ -bound (3.24), if uniform ellipticity (3.2) and uniform regularity (3.4) is replaced by the weaker conditions (3.6) and (3.7) or even  $a \in L^{2(1+\delta)}(\Omega; W^{1,\infty}(D))$  with  $\delta > 0$ , respectively.

**Remark 3.10.** The space  $W$  can be characterized as a weighted Sobolev space with weights vanishing at vertices and (in case  $d = 3$ ) at edges of the polyhedron  $D$  (see, e.g., [24]).

#### 4. MULTI-LEVEL MONTE-CARLO FINITE ELEMENT METHOD

In the following section, we first introduce continuous, piecewise linear Finite Element discretizations of the pathwise random obstacle problems (3.14) with constraints  $K$ , bilinear form  $b(\omega; \cdot, \cdot)$ , and right hand side  $f$  defined in (3.16) and (3.17), respectively. Under suitable regularity assumptions we state an optimal error estimate that holds uniformly for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . Together with well-known convergence results on Monte-Carlo (MC) sampling [8] this is the main tool for the convergence and complexity analysis of Monte-Carlo and Multi-Level-Monte-Carlo Finite Element Methods (MC-FEM and MLMC-FEM for short) for stochastic mean-square obstacle problems of the form (3.20). In the complexity analysis we assume that almost optimal algebraic solvers for the iterative solution of the discrete pathwise obstacle problems are available. Later in Section 4.5, we show that Monotone Multi-Grid (MMG) methods [4, 29, 31] have this property at least for problems in space dimensions  $d = 1, 2$ . As a consequence, as in the unconstrained case, the resulting Multi-Level Monte-Carlo Finite Element Method with algebraic MMG solver (MLMC-MMG-FEM) for stochastic obstacle problems achieve almost optimal complexity.

**4.1. Galerkin Finite Element Approximation.** Throughout this section we assume that  $D$  is a polyhedral domain, for simplicity. We consider a sequence of partitions  $\{\mathcal{T}_l\}_{l \geq 0}$  of  $D$  into simplices as resulting from uniform refinements of a coarse, regular simplicial partition  $\mathcal{T}_0$  (see, e.g., [9, 11] for details). By construction, the sequence  $\{\mathcal{T}_l\}_{l \geq 0}$  is shape regular (cf., e.g., [12, 13]) and the mesh width  $h_l$ ,

$$h_l = \max\{\text{diam}(T) : T \in \mathcal{T}_l\} ,$$

of  $\mathcal{T}_l$  satisfies  $h_l = \frac{1}{2} h_{l-1} = 2^{-l} h_0$ . For each refinement level  $l = 0, 1, \dots$  we define a corresponding sequence of nested Finite Element spaces

$$(4.1) \quad V_0 \subset V_1 \subset \dots \subset V_l \subset \dots$$

with  $V_l$  given by

$$(4.2) \quad V_l = \{v \in V : v|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_l\}.$$

Here,  $\mathcal{P}_1(T) = \text{span}\{x^\alpha : |\alpha| \leq 1\}$  denotes the space of linear polynomials on  $T$  so that  $V_l$  consists of all continuous functions on  $D$ , which are piecewise linear on all  $T \in \mathcal{T}_l$  and satisfy homogenous Dirichlet boundary conditions. The dimension  $N_l$  of  $V_l$  agrees with the number of elements of the set  $\mathcal{N}_l$  of interior nodes of  $\mathcal{T}_l$ . Assuming  $\chi \in H^2(D)$  (cf. Assumption 3.1) and utilizing  $H^2(D) \subset C(\bar{D})$  for  $d = 1, 2, 3$  space dimensions, we then may define

$$(4.3) \quad K_l := \{v \in V_l : v(p) \geq \chi(p) \quad \forall p \in \mathcal{N}_l\}, \quad \mathcal{K}_l := \overline{L^\infty(\Omega; K_l)}^{\|\cdot\|_V}.$$

Note that  $K_l \notin \mathcal{K}$  and thus  $\mathcal{K}_l \notin \mathcal{K}$ , in general, but the sets  $K_l \subset V_l$  and  $\mathcal{K}_l \subset \mathcal{V}_l$  still are norm-closed, convex cones in these spaces. Under Assumption 3.1 on the obstacle  $\chi$ , the cones  $K_l$  and  $\mathcal{K}_l$  have common vertex 0, i.e.,  $0 \in K_l$  and  $0 \in \mathcal{K}_l$  holds for all  $l = 0, 1, 2, \dots$  so that Proposition 2.3 is applicable.

Now the corresponding discretization of the mean-square weak formulation (3.20) of the stochastic elliptic boundary value problem (1.4) reads

$$(4.4) \quad u_l \in \mathcal{K}_l : \quad B(u_l, v_l - u_l) \geq F(v_l - u_l) \quad \forall v_l \in \mathcal{K}_l.$$

If Assumptions 3.1 and 3.2 a) hold, then, by Theorem 3.6, the discrete sEVI (4.4) has a unique stochastic Finite Element solution  $u_l \in L^2(\Omega; \mathcal{K}_l)$  for all  $l = 0, 1, 2, \dots$ .

Rather than the mean-square projection (4.4) MC and MLMC methods will be based on the pathwise Finite Element discretization

$$(4.5) \quad u_l(\omega) \in K_l : \quad b(\omega; u_l(\omega), v_l - u_l(\omega)) \geq \ell(v_l - u_l(\omega)) \quad \forall v_l \in K_l, \quad \omega \in \Omega,$$

of the pathwise sEVI (3.14). In particular, for each draw of  $a(\cdot, \omega)$ , a discrete deterministic obstacle problem of the form (4.5) will have to be solved in MC and MLMC sampling. As in Proposition 3.4, existence, uniqueness, and upper bounds follow from Theorem 3.3 and Proposition 2.3, respectively.

**Proposition 4.1.** *Let Assumption 3.1 hold. Then there exists a unique solution  $u_l(\omega) \in K_l$  of (4.5) for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  and  $l \in \mathbb{N}$  that satisfies*

$$(4.6) \quad \sup_{l \in \mathbb{N}} \|u_l(\omega)\|_V \leq \frac{1}{\check{a}(\omega)} \|f(\omega)\|_{L^2(D)}, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

*Under the additional Assumption 3.2 a), the family of pathwise solutions  $\{u_l(\omega) : \omega \in \Omega\} \subset L^\infty(\Omega; K)$  coincides with the unique mean-square solution  $u_l \in \mathcal{K}_l$  of (4.4) and we have the a-priori estimate*

$$(4.7) \quad \sup_{l \in \mathbb{N}} \|u_l(\omega)\|_V \leq \frac{1}{a_-} \|f(\omega)\|_{L^2(D)}, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Now we are ready to show a pathwise error estimate that holds uniformly  $\mathbb{P}$ -a.e. in  $\Omega$ .

**Proposition 4.2.** *Let Assumptions 3.1 and 3.2 hold. Then there exists a positive constant  $C = C(a, f, \chi)$  independent of  $\omega \in \Omega$  and  $l \in \mathbb{N}$ , such that*

$$(4.8) \quad \|u(\omega) - u_l(\omega)\|_V \leq Ch_l, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

*Proof.* We define the representation  $A(\omega) : V \rightarrow V^*$  of  $b(\omega; \cdot, \cdot)$  via the Riesz representation theorem by  $\langle A(\omega)w, v \rangle = b(\omega; w, v) \forall v, w \in V$ , where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing of  $V^*$  and  $V$ . By Proposition 3.7, Assumptions 3.1 and 3.2 imply  $H^2(D)$ -regularity, i.e.,

$$A(\omega)u(\omega) = \nabla \cdot (a(\omega)\nabla u(\omega)) \in L^2(D)$$

for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . Then the proof of the pathwise bound

$$(4.9) \quad \begin{aligned} & \|u(\omega) - u_l(\omega)\|_V^2 \leq \\ & \inf_{v_l \in K_l} \left( \frac{2}{\tilde{a}(\omega)} \|f(\omega) - A(\omega)u(\omega)\|_{L^2(D)} \|u(\omega) - v_l\|_{L^2(D)} + \frac{\hat{a}(\omega)^2}{\tilde{a}(\omega)^2} \|u(\omega) - v_l\|_V^2 \right) \\ & \quad + \|f(\omega) - A(\omega)u(\omega)\|_{L^2(D)} \inf_{v \in K} \|u_l(\omega) - v\|_{L^2(D)} \\ & = I + II \end{aligned}$$

follows from classical arguments due to Falk [17] (cf., e.g., [15, Thm. 5.1.1], in particular [15, (5.1.11)]). We estimate the terms  $I$  and  $II$  separately. The error estimate

$$(4.10) \quad I \leq C(a, f)h_l^2, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega$$

follows from the regularity assumption (3.3), the a-priori bound (3.27), uniform ellipticity (3.2), and well-known interpolation error estimates (cf., e.g., [15, Theorem 3.1.6]). The remaining deterministic estimate

$$II \leq C(\chi)h_l^2$$

is stated in the proof of [15, Theorem 5.1.2].  $\square$

We proceed with an analysis of the rate of convergence of the Monte-Carlo method for the solution of the stochastic elliptic problem (3.20). First we derive the estimate for the solution which is not discretized in space and then generalize this result to the Finite Element solution.

**4.2. Rate of Convergence of the Monte-Carlo Method.** We estimate the expectation  $\mathbb{E}[u] \in V$  by the mean  $E_M[u]$ ,

$$(4.11) \quad E_M[u] := \frac{1}{M} \sum_{i=1}^M u^i \in V,$$

over solution samples  $u^i \in V$ ,  $i = 1, \dots, M$ , corresponding to  $M$  independent, identically distributed realizations of the random input data  $a$  and  $f$ .

The following result is a bound on the *statistical error* resulting from this Monte-Carlo estimator.

**Lemma 4.3.** *For any  $M \in \mathbb{N}$  and for all  $u \in L^2(\Omega; V)$  we have the error estimate*

$$(4.12) \quad \|\mathbb{E}[u] - E_M[u]\|_{L^2(\Omega; V)} \leq M^{-1/2} \|u\|_{L^2(\Omega; V)}.$$

*Proof.* With the usual interpretation of the sample average  $E_M[u]$  as  $V$ -valued random variable, the independence of the identically distributed samples  $u^i$  implies

$$\begin{aligned} \|\mathbb{E}[u] - E_M[u]\|_{L^2(\Omega; V)}^2 &= \mathbb{E} \left[ \left\| \mathbb{E}[u] - \frac{1}{M} \sum_{i=1}^M u^i \right\|_V^2 \right] = \frac{1}{M^2} \sum_{i=1}^M \mathbb{E} [\| \mathbb{E}[u] - u^i \|_V^2] \\ &= \frac{1}{M} \mathbb{E} [\| \mathbb{E}[u] - u \|_V^2] = \frac{1}{M} (\mathbb{E} \|u\|_V^2 - \|\mathbb{E}[u]\|_V^2) \leq \frac{1}{M} \|u\|_{L^2(\Omega; V)}^2. \end{aligned}$$

□

**4.3. Single-Level Monte-Carlo Finite Element Method.** Monte-Carlo Finite Element methods (MC–FEM) are obtained by suitable Finite Element approximations of the ‘samples’  $u^i$  occurring in (4.11). To this end, we replace  $u^i$  by Galerkin Finite Element approximations  $u_l^i = u_l(\omega^i) \in K_l$  that can be computed from the discrete deterministic obstacle problems (4.5) with corresponding  $\omega^i$ .

The Monte-Carlo Finite Element (MC–FE) approximation of  $\mathbb{E}[u]$  thus reads

$$(4.13) \quad E_M[u_l] := \frac{1}{M} \sum_{i=1}^M u_l^i \in L^2(\Omega; V_l).$$

**Remark 4.4.** The MC–FE approximation is conforming in the sense that  $E_M[u_l] \in \mathcal{K}_l$  holds for all  $M \in \mathbb{N}$ , because  $K_l$  is convex, and  $E_M[u_l]$  is a convex combination of elements  $u_l^i \in K_l$ .

We now establish a first error estimate for the MC–FE method.

**Theorem 4.5.** *Let Assumptions 3.1 and 3.2 hold. Then we have the error bound*

$$(4.14) \quad \|\mathbb{E}[u] - E_M[u_l]\|_{L^2(\Omega; V)} \leq C(a, f, \chi) \left( M^{-\frac{1}{2}} + h_l \right).$$

*Proof.* We split the left hand side of the above estimate as follows

$$\begin{aligned} \|\mathbb{E}[u] - E_M[u_l]\|_{L^2(\Omega; V)} &\leq \|\mathbb{E}[u] - \mathbb{E}[u_l]\|_{L^2(\Omega; V)} + \|\mathbb{E}[u_l] - E_M[u_l]\|_{L^2(\Omega; V)} \\ &\leq \mathbb{E}[\|u - u_l\|_V] + \|\mathbb{E}[u_l] - E_M[u_l]\|_{L^2(\Omega; V)}. \end{aligned}$$

The first term on the right hand side is bounded according to Proposition 4.2 and the second term is estimated according to Lemma 4.3. □

The optimal choice of sample size  $M$  versus grid size  $h_l$  for a fixed error is obtained by equilibrating the statistical and the discretization error in (4.14). Hence, Theorem 4.5 yields the basic relation

$$(4.15) \quad M = O(h_l^{-2}) = O(2^{2l}).$$

We now provide an upper bound for the computational cost of the MC–FEM (4.13) under the following assumption, that discrete obstacle problems of the form (4.5) can be solved up to discretization accuracy in near optimal complexity, which will be verified in what follows.

**Assumption 4.6.** Approximations  $\tilde{u}_l(\omega)$  of solutions  $u_l(\omega)$  of deterministic obstacle problems of the form (4.5) which provide the error bound

$$\|u_l(\omega) - \tilde{u}_l(\omega)\|_V \leq Ch_l, \quad l = 0, 1, \dots,$$

can be evaluated at computational cost which is bounded, as  $l \rightarrow \infty$ , by  $O((1+l)^\nu N_l)$  with the constant implied in  $O(\cdot)$  being independent of  $l$ .

**Theorem 4.7.** *Consider some fixed  $L \in \mathbb{N}$  and let Assumptions 3.1, 3.2, and 4.6 hold. Then the approximation*

$$E_M[u_L] = \frac{1}{M_L} \sum_{i=1}^M \tilde{u}_L^i, \quad M = O(2^{2L}),$$

of  $\mathbb{E}[u]$  with accuracy

$$\|\mathbb{E}[u] - E_M[u_L]\|_{L^2(\Omega; V)} = O(h_L)$$

can be evaluated at the computational cost which is bounded, as  $L \rightarrow \infty$ , by

$$(4.16) \quad O((1+L)^\nu N_L^s), \quad \text{with } s = \begin{cases} 3, & d=1 \\ 2, & d=2 \\ 1 + \frac{2}{3}, & d=3 \end{cases}.$$

*Proof.* The assertions follow from the discretization error estimate in Theorem 4.5 and the basic relations (4.15).  $\square$

**4.4. Multi-Level Monte-Carlo Finite Element Method.** Instead of approximating  $E(u)$  directly, the Multi-Level Monte-Carlo Finite Element (MLMC-FE) method is based on suitable approximations of increments on the levels  $l = 1, \dots, L$  of the hierarchy (4.1).

With the notation  $u_0 := 0$ , we may write

$$u_L = \sum_{l=1}^L (u_l - u_{l-1}).$$

The linearity of the expectation operator  $\mathbb{E}[\cdot]$  yields

$$(4.17) \quad \mathcal{K}_L \ni \mathbb{E}[u_L] = \mathbb{E} \left[ \sum_{l=1}^L (u_l - u_{l-1}) \right] = \sum_{l=1}^L \mathbb{E}[u_l - u_{l-1}].$$

In the MLMC-FE method, we estimate  $\mathbb{E}[u_l - u_{l-1}]$  by a level dependent number  $M_l$  of samples, i.e. we estimate  $\mathbb{E}[u]$  by the MLMC estimator

$$(4.18) \quad E^L[u_L] := \sum_{l=1}^L E_{M_l}[u_l - u_{l-1}].$$

**Remark 4.8.** We emphasize that, in contrast to the classical MC-FE method, the MLMC-FE method is non-conforming, i.e.,  $E^L[u_L] \notin \mathcal{K}_L$ , in general. The reason is that

$$E^L[u_L] = E_{M_L}[u_L] + \sum_{l=1}^{L-1} (E_{M_l}[u_l] - E_{M_{l+1}}[u_l])$$

with  $E_{M_L}[u_L] \in \mathcal{K}_L$ , but  $E_{M_l}[u_l] - E_{M_{l+1}}[u_l] \neq 0$ , in general.

The convergence of the MLMC-FE method is guaranteed by the following result.

**Theorem 4.9.** *Let Assumptions 3.1 and 3.2 hold. Then the MLMC-FE approximation  $E^L[u_L]$  defined in (4.18) of the expectation  $\mathbb{E}[u]$  of the solution  $u \in L^2(\Omega; W)$  to the mean-square weak formulation (3.20) of the sEVI (1.4) admits the error bound*

$$(4.19) \quad \|\mathbb{E}[u] - E^L[u_L]\|_{L^2(\Omega; V)} \leq C(a, f, \chi) \left( h_L + \sum_{l=1}^L h_l M_l^{-1/2} \right).$$

*Proof.* We rewrite the error to be estimated as in the proof of Theorem 4.5 according to

$$\begin{aligned} \|\mathbb{E}[u] - E^L[u_L]\|_{L^2(\Omega; V)} &= \|\mathbb{E}[u] - \mathbb{E}[u_L] + \mathbb{E}[u_L] - \sum_{l=1}^L E_{M_l}[u_l - u_{l-1}]\|_{L^2(\Omega; V)} \\ &\leq \|\mathbb{E}[u] - \mathbb{E}[u_L]\|_{L^2(\Omega; V)} + \left\| \sum_{l=1}^L (\mathbb{E}[u_l - u_{l-1}] - E_{M_l}[u_l - u_{l-1}]) \right\|_{L^2(\Omega; V)} \\ &=: I + II. \end{aligned}$$

We calculate the error bounds for the terms  $I$  and  $II$  separately.

Term I: By Jensen's and by the Cauchy-Schwarz inequality, we obtain the bound

$$I \leq (\mathbb{E}[\|u - u_l\|_V^2])^{1/2} = \|u - u_l\|_{L^2(\Omega;V)} \leq C(a, f, \chi) h_l$$

for every  $l = 1, \dots, L$ . In particular for  $l = L$  we obtain the asserted bound for Term  $I$ .

Term II: By the triangle inequality, we must consider for each  $l = 1, \dots, L$  the term

$$\|\mathbb{E}[u_l - u_{l-1}] - E_{M_l}[u_l - u_{l-1}]\|_{L^2(\Omega;V)} .$$

Each of these terms is estimated as follows.

$$\begin{aligned} \|\mathbb{E}[u_l - u_{l-1}] - E_{M_l}[u_l - u_{l-1}]\|_{L^2(\Omega;V)} &= \|(\mathbb{E} - E_{M_l})[u_l - u_{l-1}]\|_{L^2(\Omega;V)} \\ &\leq M_l^{-1/2} \|u_l - u_{l-1}\|_{L^2(\Omega;V)} \\ &\leq M_l^{-1/2} (\|u - u_l\|_{L^2(\Omega;V)} + \|u - u_{l-1}\|_{L^2(\Omega;V)}) \\ &\leq C(a, f, \chi) M_l^{-1/2} (h_l + h_{l-1}) \\ &= 3C(a, f, \chi) h_l M_l^{-1/2} . \end{aligned}$$

Here we used Lemma 4.3 and Proposition 4.2. Summation of these estimates from  $l = 1, \dots, L$  completes the proof.  $\square$

The preceding result gives an error bound for the MLMC-FE approximation, for *any* distribution  $\{M_l\}_{l=1}^L$  of samples over the mesh levels. As in the single-level Monte-Carlo approximation the key question to be answered by our error analysis is the relation of mesh-width versus sample size, in order to retain the asymptotic rate of convergence  $O(h_l)$  from the deterministic case, while minimizing the overall work.

**Theorem 4.10.** *Let Assumptions 3.1, 3.2, and 4.6 hold. Then the MLMC-FE approximation  $E^L[u_L]$  defined in (4.18) of the expectation  $\mathbb{E}[u]$  of the solution to the mean-square weak formulation (3.20) of the sEVI (1.4) with the number  $M_l$  of MC samples on mesh refinement level  $l$  given by*

$$(4.20) \quad M_l = O(l^{2+2\varepsilon} 2^{2(L-l)} h_0), \quad l = 1, 2, \dots, L,$$

with  $\varepsilon > 0$  admits the error bound

$$\|\mathbb{E}[u] - E^L[u_L]\|_{L^2(\Omega;V)} \leq Ch_L \|f\|_{L^2(\Omega;L^2(D))}$$

and can be evaluated at computational cost which is bounded, as  $L \rightarrow \infty$ , by

$$(4.21) \quad O(C_{L,d} N_L), \quad \text{with } C_{L,d} = \begin{cases} (1+L)^{2+\nu+2\varepsilon} N_L, & d=1 \\ (1+L)^{3+\nu+2\varepsilon}, & d=2 \\ (1+L)^{2+\nu+2\varepsilon}, & d=3 \end{cases} .$$

*Proof.* The convergence result in Theorem 4.9 suggests that we choose  $M_l$  such that the overall rate of convergence is  $O(h_L)$ . With the choice

$$(4.22) \quad M_l = O(l^{2+2\varepsilon} (h_l/h_L)^2) = O(l^{2+2\varepsilon} 2^{2(L-l)}), \quad l = 1, \dots, L,$$

for some  $\varepsilon > 0$ , we obtain from (4.19) the asserted error bound, since for  $\varepsilon > 0$  this implies

$$\begin{aligned} \sum_{l=1}^L h_l M_l^{-1/2} &\leq C \sum_{l=1}^L 2^{-l} l^{-(1+\varepsilon)} 2^{(l-L)} h_0 \leq C 2^{-L} h_0 \sum_{l=1}^L l^{-(1+\varepsilon)} \\ &\leq C h_L \sum_{l=1}^L l^{-(1+\varepsilon)} = C(\varepsilon) h_L. \end{aligned}$$

It remains to estimate the computational cost. Utilizing Assumption 4.6 and (4.20), the cost is  $O((1+l)^\nu N_l M_l)$  on each level  $l$ . There results the following upper bound for the overall computational cost

$$\sum_{l=1}^L O((1+l)^\nu N_l M_l) \leq C \sum_{l=1}^L (1+l)^\nu 2^{dl} l^{2+2\varepsilon} 2^{2(L-l)} \leq C(1+L)^{2+\nu+2\varepsilon} N_L \sum_{l=1}^L 2^{(d-2)(l-L)}.$$

This proves the assertion.  $\square$

**4.5. Algebraic Solution.** We now discuss the evaluation of approximations  $\tilde{u}_l(\omega)$  of Finite Element solutions  $u_l(\omega)$  of deterministic obstacle problems of the form (4.5) by iterative solvers.

**Assumption 4.11.** There is an iterative scheme  $M_l : V_l \rightarrow V_l$  for the approximate solution of deterministic obstacle problems of the form (4.5) with symmetric bilinear form  $b(\omega; \cdot, \cdot)$ ,  $\omega \in \Omega$ , such that  $M_l v$  can be evaluated with optimal computational cost  $O(N_l)$  and such that

$$(4.23) \quad J(M_l v) - J(u_l(\omega)) \leq (1 - c_0(1+l)^{-(\nu-1)}) (J(v) - J(u_l(\omega)))$$

holds with some constants  $c_0 > 0$  and  $\nu \geq 1$ , independent of  $v \in V_l$ , of  $\omega$ , and of  $l = 0, 1, \dots$

**Remark 4.12.** Assumption 4.11 is fulfilled by various multigrid methods. Tai [40] proved logarithmic upper bounds of the form (4.23) with  $\nu = 3$  for a class of Subset Decomposition methods. Badea [3] showed (4.23) with  $\nu = 6$  for a projected multi-level relaxation scheme [23, Section 5.1]. Badea [4] recently extended these results to obtain  $\nu = 5$  for Standard Monotone Multigrid (STDMMG) [23, Section 5.2] which, of these three multigrid methods, is the most efficient one. All these results are restricted to  $d = 2$  space dimensions but do not require additional regularity of the exact solution.

**Proposition 4.13.** *Assume that an initial approximation  $\tilde{u}_0(\omega) \in V_0$  with the property  $\|u_0(\omega) - \tilde{u}_0(\omega)\|_V \leq C_0 h_0$  with some constant  $C_0 > 0$  (depending on the data) is given and that Assumption 4.11 holds true. Then Assumption 4.6 is satisfied.*

*Proof.* Exploiting that the Finite Element spaces are nested, (4.1), we inductively compute a sequence of approximations  $\tilde{u}_i(\omega) \in V_i$ ,  $i = 0, \dots, l$ . To this end, starting with the given  $\tilde{u}_0 \in V_0$ , we determine  $\tilde{u}_i(\omega) \in V_i$  from the given  $\tilde{u}_{i-1}(\omega) \in V_{i-1}$  on the previous level  $i-1$  as follows. If

$$(4.24) \quad \|u_i(\omega) - \tilde{u}_{i-1}(\omega)\|_V \leq C_0 2^{-i},$$

then we simply set  $\tilde{u}_i = \tilde{u}_{i-1}$ . Otherwise, the approximation  $\tilde{u}_i(\omega) = M_i^{k_i} \tilde{u}_{i-1}(\omega)$  is computed by  $k_i$  applications of the iterative solver  $M_i$  to  $\tilde{u}_{i-1}(\omega)$ , where  $k_i$  is chosen such that the stopping criterion

$$(4.25) \quad \|u_i(\omega) - M_i^{k_i} \tilde{u}_{i-1}(\omega)\|_V \leq \frac{\sigma}{2} \|u_i(\omega) - \tilde{u}_{i-1}(\omega)\|_V$$

holds with some fixed  $\sigma < 1$ . This process is referred to as *nested iteration* (see, e.g., [26, Chapter 5]) or *full multigrid*. In the case that (4.24) holds, we obviously have

$$(4.26) \quad \|u_i(\omega) - \tilde{u}_i(\omega)\|_V \leq C_0 2^{-i}.$$

We assume without loss of generality that  $i_0 = 0$  is the largest level  $i_0 \leq l$  such that (4.26) holds true which means that (4.24) does not occur.

Utilizing (4.25), we compute

$$\begin{aligned} \|u(\omega) - \tilde{u}_l(\omega)\|_V &\leq \|u(\omega) - u_l(\omega)\|_V + \|u_l(\omega) - \tilde{u}_l(\omega)\|_V \\ &\leq \|u(\omega) - u_l(\omega)\|_V + \frac{\sigma}{2} \|u_l(\omega) - \tilde{u}_{l-1}(\omega)\|_V \\ &\leq (1 + \frac{\sigma}{2}) \|u(\omega) - u_l(\omega)\|_V + \frac{\sigma}{2} \|u(\omega) - \tilde{u}_{l-1}(\omega)\|_V \\ &\leq (\frac{\sigma}{2})^l \|u_0(\omega) - \tilde{u}_0(\omega)\|_V + (1 + \frac{\sigma}{2}) \sum_{i=0}^l (\frac{\sigma}{2})^i \|u(\omega) - u_{l-i}(\omega)\|_V \end{aligned}$$

Exploiting  $\|u(\omega) - \tilde{u}_0(\omega)\|_V \leq C_0 h_0$ ,  $h_l = 2^{i-l} h_i$  for  $0 \leq i \leq l$ , and the discretization error estimate (4.8), we obtain the norm error estimate in Assumption 4.6.

We now estimate the computational cost. No computations are needed in the (non-generic) case (4.24). Hence, we assume

$$(4.27) \quad \|u_i(\omega) - \tilde{u}_{i-1}(\omega)\|_V > C_0 2^{-i}.$$

Utilizing (2.6) and the equivalence of the energy norm  $\|\cdot\|_b = b(\omega; \cdot, \cdot)^{1/2}$  and the canonical norm  $\|\cdot\|_V$  together with the upper bound (4.23) for the convergence rate of the energy error, and the upper bound (4.8) for the discretization error, it turns out that the stopping criterion (4.25) is satisfied, if  $k_i$  is chosen such that

$$c(1 - c_0(1+i)^{-(\nu-1)})^{k_i} \leq \frac{\sigma^2}{4} \|u_i(\omega) - \tilde{u}_i(\omega)\|_V$$

holds with a suitable positive constant  $c$  which is independent of  $i$  and of  $\nu$ . In the light of (4.27), it is sufficient to choose  $k_i$  according to

$$k_i \geq (\log(2)i - \log(C_0/c)) / -\log(1 - c_0(1+i)^{-(\nu-1)}) \geq C(1+i)^\nu.$$

Hence, the computational cost on each level  $i$  is bounded by  $O((1+i)^\nu N_i)$ . Utilizing  $N_i = O(2^{i-l} N_l)$ , an upper bound for the overall computational cost is given by

$$(4.28) \quad \sum_{i=1}^l O((1+i)^\nu N_i) = O((1+l)^\nu N_l).$$

□

Note that the additional power of  $1+l$  is caused by the mismatch between the lower and upper bound in (2.6). As the initial grid is intentionally coarse, suitable initial approximations  $\tilde{u}_0(\omega)$  of  $u_0(\omega)$  and thus of  $u(\omega)$  can be often obtained by methods for complementarity problems with moderate size.

The main result of this section is an immediate consequence of Theorems 4.7 and 4.10 together with Remark 4.12.

**Corollary 4.14.** *Let Assumptions 3.1, 3.2 hold, assume that  $d = 2$ , and let STDMMG be used for the approximate solution of the discrete pathwise obstacle problems of the form (4.5). Then the resulting MC-MMG-FE approximation  $E_M[u]$  and the resulting MLMC-MMG-FE approximation  $E^L[u_L]$  of the expectation  $\mathbb{E}[u]$  both have optimal accuracy  $O(h_L)$  in  $L^2(\Omega; V)$  and require the computational cost  $O((1+L)^2 N_L^2)$  and  $O((1+L)^{8+2\epsilon} N_L)$ , respectively.*

Hence, utilizing the recent convergence results by Badea [4] for STDMMG in  $d = 2$  space dimensions, the order of computational cost for MLMC-FE for the approximation of the statistical mean  $\mathbb{E}[u]$  (and also for spatial correlation functions, see e.g. [33]) turns out to be asymptotically the same as for the multigrid solution of *a single instance of the deterministic problem, on the finest mesh at refinement level  $L$* , up to logarithmic terms. Numerical experiments indicate even mesh-independent convergence rates for STDMMG and for the recently introduced *Truncated Nonsmooth Newton-Multigrid* (TNNMG) [22, 23] as applied in the framework of nested iteration (see, e.g., Gräser and Kornhuber [23]). However, to the knowledge of the authors, mathematical justification of the observed performance of TNNMG is still open.

## 5. NUMERICAL EXPERIMENTS

We consider the stochastic obstacle problem (1.4) on the spatial domain  $D = (-1, 1)^d$ ,  $d = 1, 2$ , with “flat” obstacle  $\chi \equiv 0$ , and with parametric, stochastic diffusion coefficient

$$a(x, \omega) = 1 + \frac{\cos |x|^2}{10} Y_1(\omega) + \frac{\sin |x|^2}{10} Y_2(\omega),$$

and with the stochastic source term  $f$  given by

$$f(x, \omega) = \begin{cases} -8e^{2(Y_1(\omega)+Y_2(\omega))} \left( \frac{d}{2} a(x, \omega) \cdot ((4-d)|x|^2 - r^2) \right. \\ \quad \left. + (|x|^2 - r^2)|x|^2 \left( -\frac{\sin |x|^2}{10} Y_1(\omega) + \frac{\cos |x|^2}{10} Y_2(\omega) \right) \right), & |x| > r \\ 4r^2 e^{2(Y_1(\omega)+Y_2(\omega))} \left( d \cdot a(x, \omega) \cdot (-1 - r^2 + |x|^2) \right. \\ \quad \left. + (-2 - 2r^2 + |x|^2)|x|^2 \left( -\frac{\sin |x|^2}{10} Y_1(\omega) + \frac{\cos |x|^2}{10} Y_2(\omega) \right) \right), & |x| \leq r \end{cases}$$

denoting

$$r = r(Y_1(\omega), Y_2(\omega)) := 0.7 + \frac{Y_1(\omega) + Y_2(\omega)}{10},$$

and uniformly distributed random variables  $Y_1, Y_2 \sim \mathcal{U}(-1, 1)$ . Then, for given  $\omega \in \Omega$ , the exact solution of the resulting pathwise problem is given by

$$u(x, \omega) = \max\{(|x|^2 - r^2) e^{Y_1(\omega)+Y_2(\omega)}, 0\}^2, \quad x \in D.$$

The remainder of this section is devoted to a numerical comparison of the efficiency of single level MC-FE (cf. Section 4.3) and of the MLMC-FE approach (cf. Section 4.4).

To this end, each pathwise problem is discretized by finite elements as described in Section 4.1. To build up the hierarchy (4.1), we start from the coarse partition  $\mathcal{T}_0$  consisting of four intervals with length  $h_0 = 1/4$  for  $d = 1$  and  $\mathcal{T}_0$  resulting from uniform refinement of a partition of  $D$  into two congruent triangles for  $d = 2$  space dimensions. Approximate solution of the resulting discrete pathwise problems of the form (4.4) is performed by the truncated nonsmooth Newton multigrid method (TNNMG) [22, 23] with nested iteration (cf. [26, Chapter 5] and the proof of Proposition 4.13). The reason is that TNNMG is easier to implement and usually converges faster than STDMMG [23]. Denoting one step of TNNMG on refinement level  $i$  by  $M_i$ , the stopping criterion (4.25) is replaced by the verifiable condition

$$\|M_i^{k_i+1} \tilde{u}_{i-1}(\omega) - M_i^{k_i} \tilde{u}_{i-1}(\omega)\|_V \leq s \frac{\sigma}{2} \|M_i \tilde{u}_{i-1}(\omega) - \tilde{u}_{i-1}(\omega)\|_V$$

with a safety factor  $s \leq (1-q)/(1+q)$ . This condition relies on the a posteriori error estimate

$$(1-q)\|u_i(\omega) - v\|_V \leq \|M_i v - v\|_V \leq (1+q)\|u_i(\omega) - v\|_V, \quad v \in V_i,$$

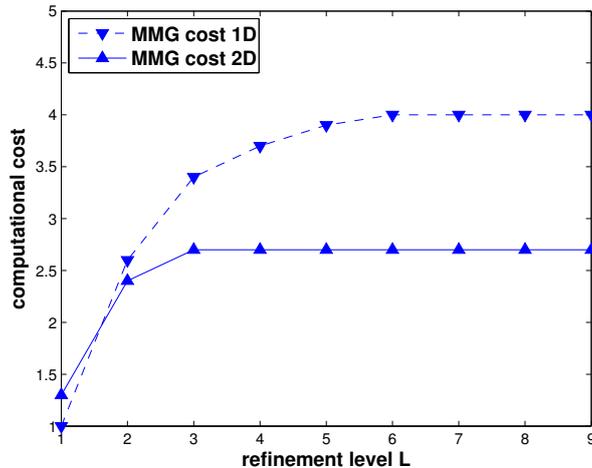


FIGURE 1. Averaged computational cost per unknown for the TNNMG multi-grid solution of obstacle problems as occurring in a MLMC computation over number of levels  $i$ .

involving the (unknown) convergence rate  $q < 1$  of  $M_i$ . We use  $s = 0.1$  and  $\sigma = \frac{1}{2}$  in our computations. Figure 1 shows the computational cost per unknown for the approximate solution of deterministic obstacle problems of the form (4.5) by TNNMG. More precisely, it shows the ratio of the average of the computational cost  $\sum_{j=1}^i k_j(\omega^i)N_j$  for all the discrete pathwise problems as occurring in the Monte Carlo computations on the levels  $i = 1, \dots, l$  of a MLMC-FE step and of the number of unknowns  $N_i$ . These (empirical) results indicate that the computational cost on level  $i$  is bounded by about  $4.0N_i$  and  $2.7N_i$  in  $d = 1$  and  $d = 2$  space dimensions, respectively. This means that, numerically, Assumption 4.6 holds with the optimal parameter  $\nu = 0$ . We obtained essentially the same results for STD-MMG. We now select the numbers of samples  $M$  and  $M_1, \dots, M_L$  to be used in MC-FE and MLMC-FE, respectively. By Theorem 4.10, the number of MC samples at mesh level  $l$ , i.e.  $M_l = \lfloor c_0 l^{2+2\varepsilon} 2^{2(L-l)} h_0 \rfloor$  (Gaussian brackets) in MLMC-FE is determined only up to positive constants  $c_0$  and  $\varepsilon$ . Instead of selecting these constants explicitly, we want to determine  $\underline{M} = (M_1, \dots, M_L) \in \mathbb{N}^L$  by minimizing the computational cost for one MLMC-FE sweep, i.e., by solving the minimization problem

$$(5.1) \quad (M_1, \dots, M_L) = \arg \min_{\underline{M} \in \mathbb{N}^L} \sum_{l=1}^L N_l M_l$$

(recall that  $\nu = 0$ ), subject to the constraints

$$(5.2) \quad 3 \sum_{l=1}^L M_l^{-\frac{1}{2}} h_l \leq h_L, \quad M_1 \geq h_L^{-2}, \quad M_l \geq 1 \quad \forall l = 1, \dots, L.$$

The particular choice  $M_l = \lfloor c_0 l^{2+2\varepsilon} 2^{2(L-l)} h_0 \rfloor$  fulfills these constraints for suitably chosen parameters  $c_0$  and  $\varepsilon$  (which choices are independent of  $L$ ,  $l$  and of  $\varepsilon$ ). Hence, (4.21) still provides an upper bound for the computational cost associated with the numbers of samples  $M_1, \dots, M_L$  obtained by our optimization procedure. We use the approximate solution of the

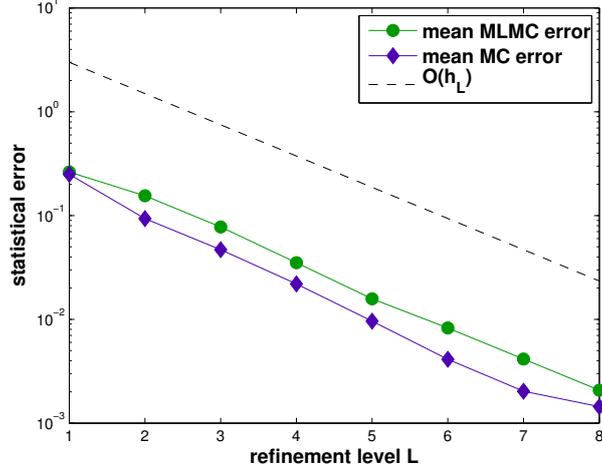


FIGURE 2. Statistical error of MC-FE and MLMC-FE over the number of refinement levels  $L$  in  $d = 1$  space dimensions.

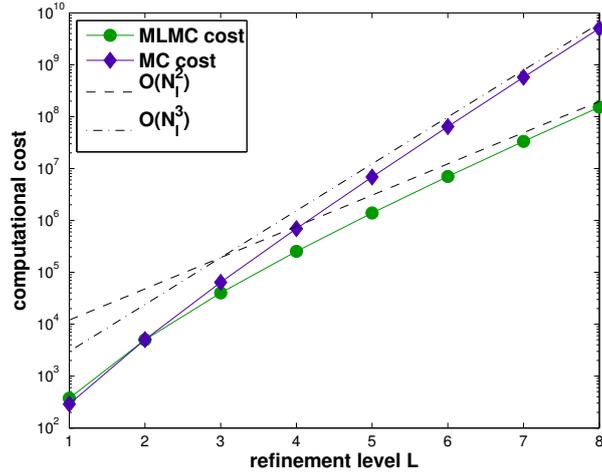


FIGURE 3. Computational cost of MC-FE and MLMC-FE over the number of refinement levels  $L$  in  $d = 1$  space dimensions.

nonlinear integer programming problem given by (5.1) and (5.2) as obtained by rounding up the solution in  $\mathbb{R}^L$  which in turn is computed numerically by a MATLAB routine.

As a consequence of Theorems 4.5 and 4.9, the conditions (5.2) and the selection of  $M = M_1 \geq h_L^{-2}$  for single-level MC-FE provide the error estimate

$$(5.3) \quad \|\mathbb{E}[u] - E_M[u_L]\|_{L^2(\Omega, V)} \leq \|\mathbb{E}[u] - E^L[u_L]\|_{L^2(\Omega, V)} \leq 2C(a, f, \chi)h_L.$$

The evaluation of the statistical error

$$\|\mathbb{E}[u_L] - E[u_L]\|_{L^2(\Omega, V)} = (\mathbb{E}[\|\mathbb{E}[u_L] - E[u_L]\|_V^2])^{1/2}$$

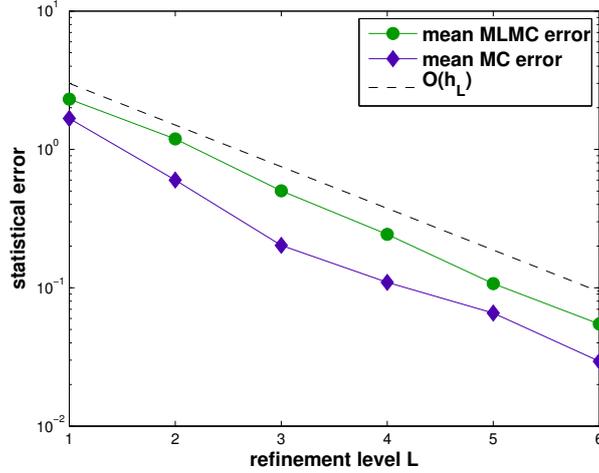


FIGURE 4. Statistical error of MC-FE and MLMC-FE over the number of refinement levels  $L$  in  $d = 2$  space dimensions.

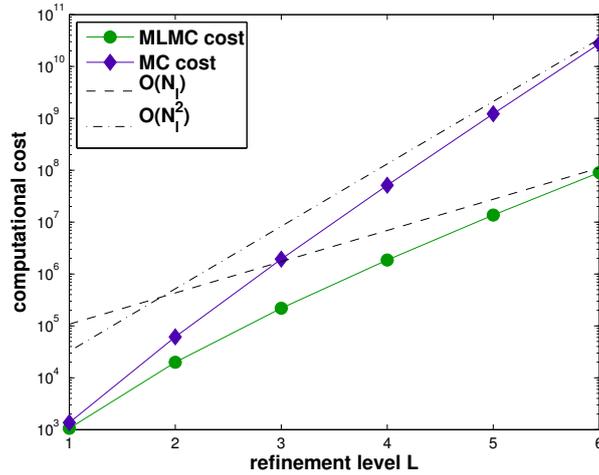


FIGURE 5. Computational cost of MC-FE and MLMC-FE over the number of refinement levels  $L$  in  $d = 2$  space dimensions.

associated with the discrete approximations  $E[u_L] = E_M[u_L]$ ,  $E^L[u_L]$  requires the evaluation of the expectation value of the random variable  $\|\mathbb{E}[u_L] - E[u_L]\|_V^2$ . We approximate  $\mathbb{E}[\|\mathbb{E}[u_L] - E[u_L]\|_V^2]$  by a Monte-Carlo method or, more precisely, by the sample average of 10 numerically computed, independent realizations of  $\|\mathbb{E}[u_L] - E[u_L]\|_V^2$ .

We now report on the numerical solution of the model problem introduced above in  $d = 1$  space dimension. Figure 2 nicely illustrates the  $O(h_L)$  behavior of the statistical errors of MC-FE ( $-\bullet-\bullet-$ ) and MLMC-FE ( $-\blacklozenge-\blacklozenge-$ ) indicated by the dashed line. As expected from the error estimate (5.3), MC-FE is slightly more accurate than MLMC-FE. The corresponding computational cost  $N_L M$  of MC-FE ( $-\bullet-\bullet-$ ) and  $\sum_{l=1}^L N_l M_l$  of MLMC-FE ( $-\blacklozenge-\blacklozenge-$ ) over the

refinement levels  $L = 1, \dots, 8$  is depicted in Figure 3. It turns out that the cost of MC-FE asymptotically behaves like  $O(N_L^3)$  (dotted line) while MLMC-FE only requires  $O(N_L^2)$  point operations (dashed line). For this moderate number of refinement levels, the logarithmic term occurring in the theoretical upper bound (4.21) for  $\nu = 0$  is not visible.

The corresponding results for  $d = 2$  space dimensions are shown in Figure 4 and Figure 5. According to Figure 4, the statistical error again behaves like  $O(h_L)$  (dashed line) and MC-FE (-●-●-) is slightly more accurate than MLMC-FE (-◆-◆-). Figure 5 indicates that the computational cost of MC-FE (-●-●-) is of order  $O(N_L^2)$  and that MLMC-FE (-◆-◆-) provides approximations with  $O(h_l)$  accuracy at optimal computational cost  $O(N_L)$ .

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