♣ *Visual Linear Algebra and Multidimensional Geometry*

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ℐn parallel coordinates (abbr. $\|$-cs) a point in $\mathbb{R}^N$ is represented by a polygonal line (polyline) Fig. 1 (left). The polylines representing points satisfying the linear relation

$$\ell_{i,j} : x_i = m_{i,j}x_j + b_{i,j} \quad \longleftrightarrow \quad \bar{\ell}_{i,j} : (\frac{j-i}{1-m_{i,j}} + (i-1), \frac{b_{j-i}}{1-m_{i,j}}) \tag{1}$$

all intersect at a single point $\bar{\ell}_{i,j}$, whose position in Cartesian coords is given above with $(i-1)$ being the distance between the $\bar{X}_1$ and $\bar{X}_i$ axes. This point can be anywhere in the xy-plane. For $N=2$ there is a *line* $\leftrightarrow$ *point* duality, showing why the parallel coordinate system must reside in the Projective $\mathbb{P}^2$ rather than the Euclidean $\mathbb{R}^2$ plane. The projection of a line $\ell \subset \mathbb{R}^N$ on the x_ix_j 2-plane is given by $\bar{\ell}_{i,j}$. Since $(N-1)$ independent projections specify ℓ, **a line is represented by the** $(N-1)$ **points** $\bar{\ell}_{i,j}$. The two indices $i.j$ are significant and must be available when they are not clear from the context. In the example seen in Fig. 1 (center) of polylines representing collinear points, several of the $\bar{\ell}_{i,i+1}$ occur between adjacent axes. An instance where **linear dependence** between linear relations is manifested as **collinearity** of points is seen on the right. As a consequence of Desargues theorem, for $i \neq j \neq k$, the points $\bar{\ell}_{ij}, \bar{\ell}_{ik}, \bar{\ell}_{jk}$ are all on a line $\bar{L}$. For $N=3$ there is a useful $\ell \leftrightarrow \bar{L}$ correspondence where $\bar{\ell}_{12}, \bar{\ell}_{13}, \bar{\ell}_{23} \in \bar{L}$.

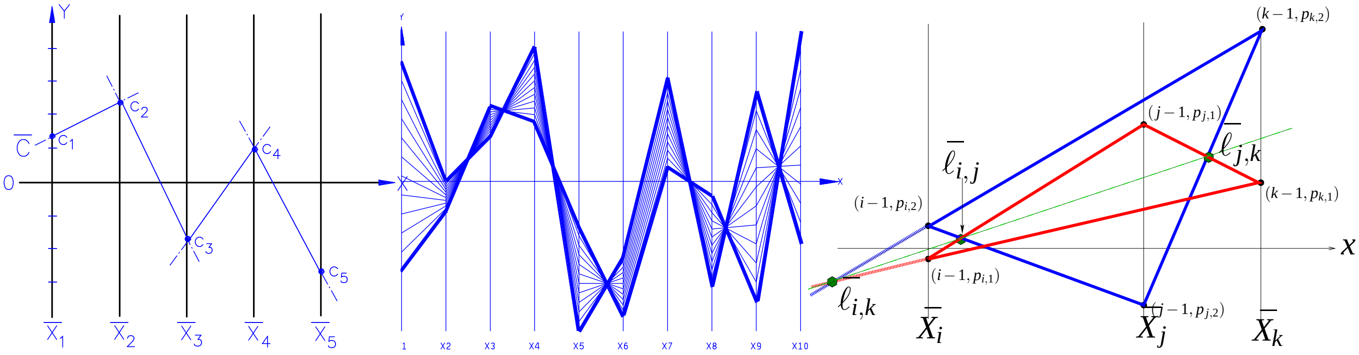

Figure 1: (Left) The polygonal line $\bar{C}$ represents the point $C = (c_1, c_2, c_3, c_4, c_5) \in \mathbb{R}^5$. (Center) Polylines representing collinear points in $\mathbb{R}^{10}$. (Right) The 3 points $\bar{\ell}_{i,j}, \bar{\ell}_{j,k}, \bar{\ell}_{i,k}$ are collinear for $i \neq j \neq k$ due to the *linear dependence* between the corresponding relations.

Coplanarity in $\|$-cs can not be recognized directly from the polylines representing points on a plane π, as seen in Fig. 2 (left), but rather from the ***lines* in that plane**. Forming lines ℓ by pairs of points on π, and obtaining the intersections $\bar{\ell}_{12}, \bar{\ell}_{13}, \bar{\ell}_{23}$ from the corresponding polylines, the $\bar{L}$ are constructed. Remarkably all the $\bar{L}$ intersect at a *single* point (first point in center) denoted by $\bar{\pi}_{123}$. A translation of the first axis $\bar{X}_1$ to $\bar{X}_{1'}$ three units to the right and repetition of the constructions for the coordinate system $\bar{X}_2, \bar{X}_3, \bar{X}_{1'}$ yields the second intersection point $\bar{\pi}_{1'23}$ with the same y-coord. For $\pi : c_1x_1 + c_2x_2 + c_3x_3 = c_0$ and $S = c_1 + c_2 + c_3$ the xy-coords are:

$$\bar{\pi}_{123} = (\frac{c_2 + 2c_3}{S}, \frac{c_0}{S}) \quad , \quad \bar{\pi}_{1'23} = (\frac{3c_1 + c_2 + 2c_3}{S}, \frac{c_0}{S}) \,. \tag{2}$$

Just these **two points represent the plane**. Their 3 indices distinguish them from the points representing lines which have two indices. As will be explained in the presentation [1] these points represent two intersecting lines in $\mathbb{R}^3$ Fig. 2 (right). There is a family of 2-flats, the **super-planes**, whose points appear in

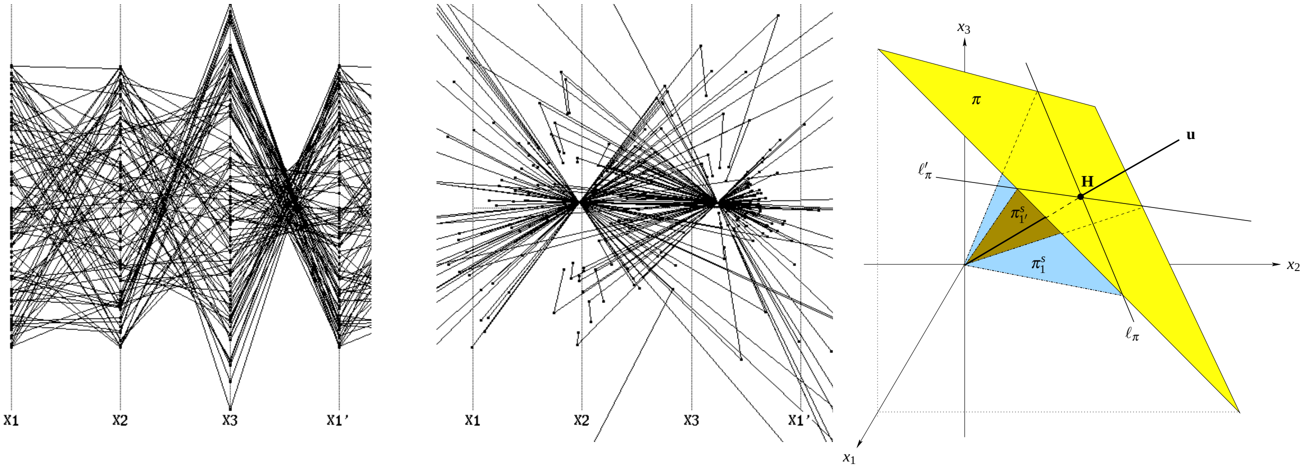

Figure 2: (Left) The polygonal lines on the first 3 axes represent points on a plane $\pi \subset \mathbb{R}^3$. No pattern is apparent. (Center) The plane π represented by just the two points. (Right) The two points represent two intersecting lines which determine π.

$\|$-cs as straight rather than polygonal lines. They have nice properties enabling us to efficiently obtain the **representation of hyperpanes in $\mathbb{R}^N$ in terms of $(N-1)$ points with $(N+1)$ indices.** In Fig. 3 (left) from the collinearity of $\bar{\ell}_{12}, \bar{\ell}_{13}, \bar{\ell}_{23}$ and $\bar{\pi}_{123}$ (i.e. $\bar{\pi}_{123} \in \bar{L}$), and similarly for $\bar{\pi}_{1'23} \in \bar{L}'$, we recognize that the line ℓ is contained in the plane π. Here, as in Fig. 1 (right), *linear dependence* is represented by *collinearity*. Among the interactive demonstrations included in the presentation is one showing the *rotation* of a plane π about a line as a *translation* of the $\bar{\pi}_{123}$ and $\bar{\pi}_{1'23}$ along $\bar{L}$ and $\bar{L}'$. Another immediate consequence is the easy construction finding the intersection of two planes $\ell = \pi \cap \rho$ (center). Namely, the representing point $\bar{\ell}_{23}$ is the intersection of the two lines through the 123 and $1'23$ points respectively. And all this generalizes nicely to higher dimensions.

This brings us to the representation of smooth hypersurfaces by their tangent hyperplanes or equivalently by their normal vectors as shown on the right for a surface $\sigma \subset \mathbb{R}^3$. The tangent plane π at each point provides the two representing points $\bar{\pi}_{123}, \bar{\pi}_{1'23}$. As a result the surface σ is represented by two planar regions $\bar{\sigma}_{123}$, $\bar{\sigma}_{1'23}$ consisting of points with the indicated indices. They are *linked* in the sense that each point $\bar{\pi}_{123} \in \bar{\sigma}_{123}$ must be matched to the correct point $\bar{\pi}_{1'23} \in \bar{\sigma}_{1'23}$ representing the tangent plane π; an example is shown in Fig. 5 (Left). This surface representation has striking properties which can be appreciated from Figs. 4, 5. From the resulting patterns surface features like folds, cusps, bumps are revealed which are usually hidden in other representations. ***Convexity*** can be recognized in any dimension from just one orientation. ***Non-orientability***, as in the Moëbius strip, can be visualized and generalized. **In effect, the**

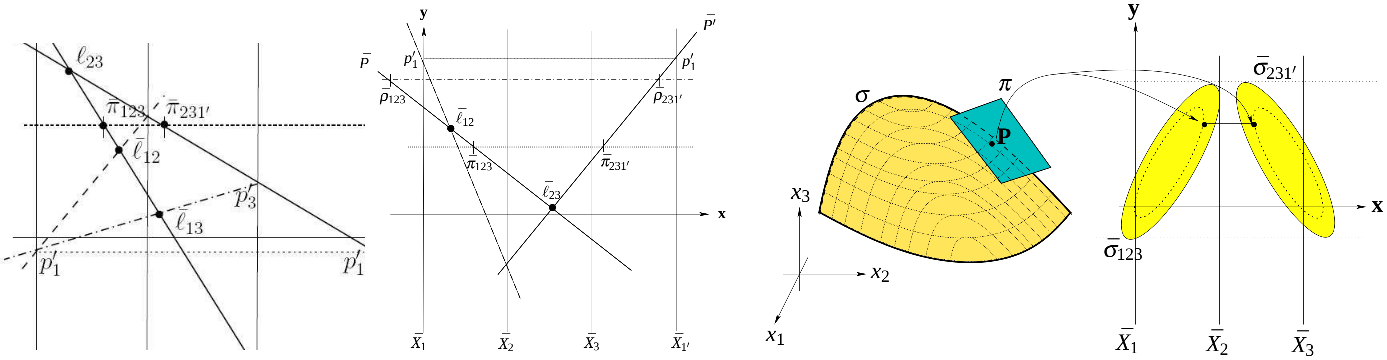

Figure 3: (Left) Line ℓ contained in plane π (Center) The intersection ℓ of two planes π and ρ. (Right) A surface $\sigma \in \mathscr{E}$ is represented by two linked planar regions $\bar{\sigma}_{123}$, $\bar{\sigma}_{231'}$.

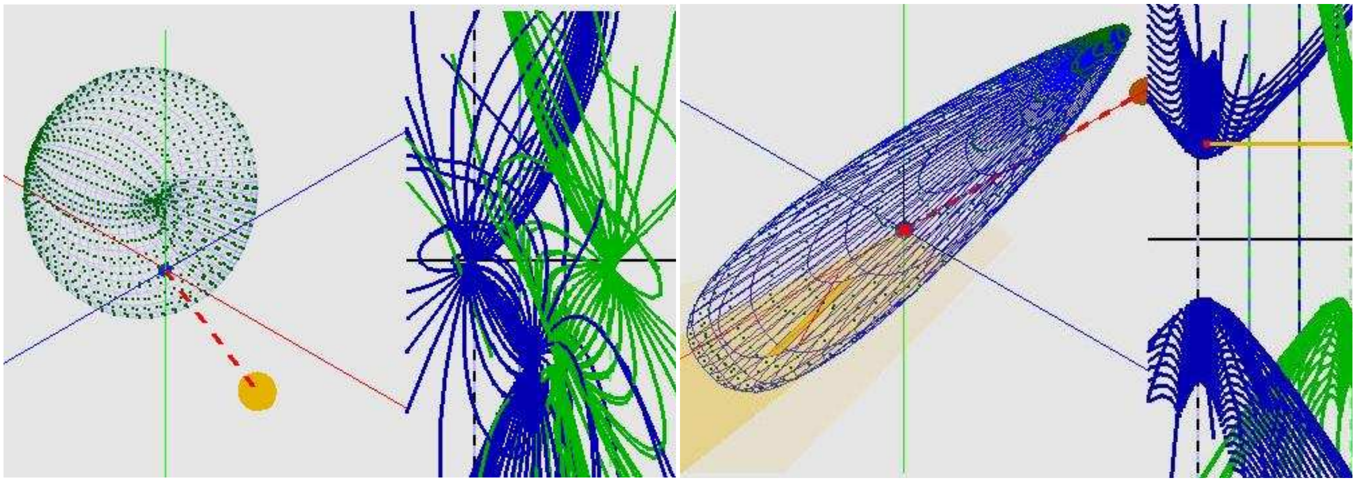

Figure 4: (Left) Surface with 2 cusps (only one is visible in the perspective) which are mapped into pairs of “swirls”; both pairs of swirsl are visible in the representation. (Right) Convex surfaces are mapped into two hyperbola-like regions (each having two assymptotes). The analogues of these surfaces in $\mathbb{R}^N$ are represented by $(N-1)$ such regions.

intuition gained from the representations in $\mathbb{R}^3$ motivates the generalizations to $\mathbb{R}^N$ yielding beautiful new dualities like: *cusp* in $\mathbb{R}^N \leftrightarrow (N-1)$ *”swirls”* in $\mathbb{R}^2$, *twist* in $\mathbb{R}^N \leftrightarrow (N-1)$ *cusps* in $\mathbb{R}^2$.

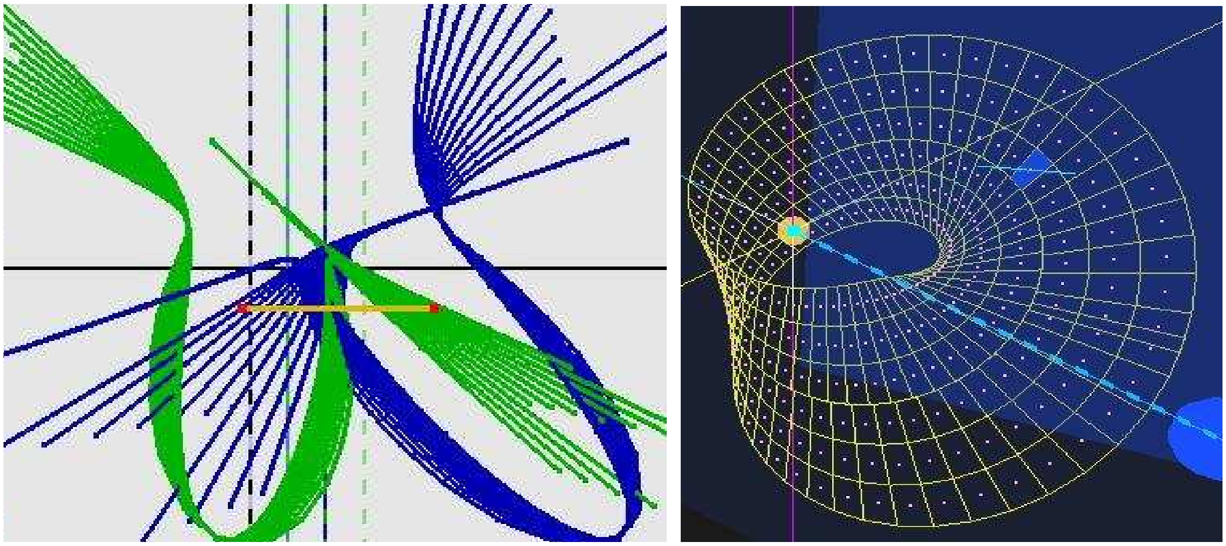

Figure 5: Möbius strip and its representation; the paired points represent a tangent plane. The two cusps on the left represent the 3-dimensional inflection point (twist). Upward and downward curves in the same direction indicate that the strip is closed. A “Möbius strip” in $\mathbb{R}^N$ is represented by $(N-1)$ similar regions with cusps.

References

[1] A. Inselberg. *Parallel Coordinates : **VISUAL** Multidimensional Geometry and its Applications*. Springer, New York, 2009.