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Swiss Federal Institute of Technology Zurich

Conservation laws with convectively filtered variables

Master Thesis

Franziska Weber

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Advisors: Prof. Dr. N.H. Risebro, Prof. Dr. R. Hiptmair
Department of Mathematics, ETH Zürich

Abstract

The properties of the solution to the convectively filtered Burgers' equation, a regularization of Burgers' equation with the convective velocity replaced by a nonlocal averaged velocity, are examined. It is found that the limit of solutions, as the regularizing parameter α goes to zero, does not satisfy an entropy inequality owing to the reversibility of the equation and the absence of an L^1-contraction estimate for the limit of solutions. In an attempt to overcome the reversibility of the equation, a model with a filter depending on time is considered. The limit of solutions turns out to be a weak solution of Burgers' equation but not the entropy solution either. Then, numerical experiments for the 1d shallow water equations are performed with methods using filtered variables. The results indicate that filtering is not an appropriate technique to regularize this system of conservation laws.

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Chapter 1

Introduction

The Euler and the Navier-Stokes equations are known to describe fundamental laws governing fluid dynamics. Even though they have been derived almost 200 years ago, they continue to pose theoretical and numerical challenges. One reason is the nonlinearity of these differential equations, which causes small scale effects, such as shocks and turbulence. They render the numerical solution of the equations extremely difficult, as one wishes on one hand to have a stable solution and on the other hand to have a good resolution of the small scale effects. It is hoped that shocks and turbulence can be modeled properly by a single technique.

Different methods have been applied in computational fluid dynamics attempting to capture the small scale structures, such as the Large Eddy simulation (LES) [8] or the Reynolds Averaged Navier-Stokes equations (RANS) [9]. Other approaches are the Lagrangian Averaged Navier-Stokes equations (LANS-α) [17] and Leray turbulence modeling [5]. In these methods an averaged convective velocity is used in the nonlinear term of the Navier-Stokes equations.

A similar quadratic nonlinear term as in the Navier-Stokes equations can be encountered in Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{1.1a}$$

$$u(x,0) = u^0(x), \quad x \in \mathbb{R}, \tag{1.1b}$$

which is the simplest example of a nonlinear conservation law and often serves as a model for more complicated equations. This equation has been well studied and its solution is known to form shocks. Even for smooth initial data, smooth solutions cease to exist after a finite period of time. For this reason, the concept of *weak solutions* has been developed: Instead of requiring Equation (1.1) to be satisfied pointwise, like a classical solution does, one seeks for solutions u satisfying the differential equation in a *weak/distributional sense*. That is, we require

$$-\int_0^T \int_{\mathbb{R}} u\varphi_t + \frac{u^2}{2}\varphi_x \, dx\, dt + \int_{\mathbb{R}} u^0(x)\varphi(x,0)\, dx = 0, \quad \forall \varphi \in C_0^1(\mathbb{R}\times[0,T)).$$

Since weak solutions are not necessarily unique, additional admissibility criteria have to be imposed to obtain uniqueness. This can be achieved by restricting to weak solutions satisfying an entropy condition:

$$\eta(u)_t + q(u)_x \leq 0, \tag{1.2}$$

where η, the entropy, is a convex function and q is the entropy flux, which satisfies $q'(u) = u\eta'(u)$. The unique weak solution for which (1.2) holds is called the *entropy solution*. The entropy solution to Burgers' equation satisfies inequality (1.2) for any convex function η. Moreover, it can be obtained as the limit of solutions $(u^\varepsilon)_{\varepsilon>0}$ of the viscous regularization

$$u_t^\varepsilon + \left(\frac{u^{\varepsilon\,2}}{2}\right)_x = \varepsilon\, u_{xx}^\varepsilon,$$

as $\varepsilon \to 0$. Due to the dissipative term on the right-hand side of the equation, the solution u^ε does not have discontinuities. In fact, it can be shown that it is infinitely differentiable for all times $t > 0$ if the initial data u^0 is in $L^\infty(\mathbb{R})$, [10].

Adding a viscous term is not the only way of regularizing Burgers' equation. Another well-known example of a regularized equation is the Korteweg de Vries equation,

$$u_t^\varepsilon + \left(\frac{u^{\varepsilon\,2}}{2}\right)_x = -\varepsilon\, u_{xxx}^\varepsilon.$$

It has been shown that solutions to this dispersive partial differential equation have the same regularity as the initial data [21], however, oscillations arise as $\varepsilon \to 0$ and the weak limit $u^\varepsilon \rightharpoonup u$ is not even a weak solution of Burgers' equation (1.1), [13, 14]. Another regularization, in fact the one we are going to investigate in this thesis, is the *convectively filtered Burgers' (CFB) equation*

$$u_t^\alpha + \overline{u}^\alpha u_x^\alpha = 0, \tag{1.3a}$$
$$\overline{u}^\alpha = g_\alpha * u^\alpha, \tag{1.3b}$$

where $g_\alpha(x) = \frac{1}{\alpha} g(\frac{x}{\alpha})$ and we assume g to be a nonnegative, symmetric and non-increasing (with respect to the absolute value of the argument) function with $\int g = 1$. In contrast to the beforehand mentioned equations, this equation is not regularized through the addition of a higher order derivative term, but by replacing the velocity u^α by the nonlocal averaged velocity $\overline{u}^\alpha$. Thanks to the convolution of u^α with g_α, $\overline{u}^\alpha$ gains additional regularity. Thus the non-conservative product uu_x of a discontinuous function with a measure, obtained when expanding the term $(u^2/2)_x$ in Burgers' equation, is replaced by the product $\overline{u}^\alpha u_x^\alpha$ which is at least well defined. It is hoped that, by analyzing the convectively filtered Burgers' equation, more insight can be gained into the more complicated model of the LANS-α equations, as they obviously share the nonlocal velocity term. Imminent questions that arise are:

- Is (1.3) locally and globally well posed?
- Does u^α converge to a limit u as $\alpha \to 0$, and is this limit a weak solution of Burgers' equation (1.1)?
- If the limit u exists and is a weak solution of Burgers' equation, is it the entropy solution of Burgers' equation?

Recent results on the CFB equation

Global well-posedness of the convecitvely filtered Burgers' equation (1.3) has been shown for a wide class of inital data and filters g in [2]. In addition, the authors Bhat and Fetecau compute explicit solutions of Riemann problems and show that for Riemann problems with left state u_L greater than the right state u_R, the CFB equation captures the behavior of the inviscid Burgers' equation, whereas for Riemann problems with $u_L < u_R$ the solutions of the regularized equation converge to weak solutions with a nonphysical shock. Convergence to a weak solution of Burgers' equation has been proved for filters g whose Fourier transform satisfies

$$\hat{g}(k) = \frac{1}{1 + \sum_{j=1}^{n} C_j k^{2j}}, \quad n < \infty,\, C_j \geq 0,\, C_n \neq 0, \tag{1.4}$$

[20]. In the same paper, the authors conjecture convergence to the entropy solution for any continuous initial data and give physical reasons why it might be true. However, to my knowledge, there is no rigorous mathematical proof of this conjecture yet. Only for the particular case of the Helmholtz filter, that is, (1.4) for $n = 1$ and $C_1 = 1$, convergence to an entropy solution has been proved under the assumption that the CFB equation is additionally regularized by a diffusion term,

$$u_t^{\alpha,\varepsilon} + \overline{u}^{\alpha,\varepsilon} u_x^{\alpha,\varepsilon} = \varepsilon\, u_{xx}^{\alpha,\varepsilon},$$

and $\alpha = o(\varepsilon)$ [6].

For the particular case of the regularization by the Helmholtz filter, Bhat and Fetecau [1] have found in addition that the CFB equation has a Hamiltonian structure.

The idea of using a filtered velocity is not new, it is Leray who proposed it for the first time in the context of the Navier-Stokes equations as early as 1934 [15]. By replacing the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ by $\overline{\mathbf{u}} \cdot \nabla \mathbf{u}$ in the equations, he intended to prove the existence of a solution to the modified equation and then to show that it converges to a weak solution of the Navier-Stokes equations when letting $\alpha \to 0$.

Outline of the thesis

In Chapter 2 we state some general properties of the solution of the convectively filtered Burgers' equation and explain the difficulties we encountered when attempting to prove convergence to the entropy solution. Since the time reversibility of the equation excludes the existence of an entropy inequality for the limit u of solutions u^α as $\alpha \to 0$, we investigate the CFB equation with a filter depending on time in Chapter 3. We have found that the limit function u is a weak solution of Burgers' equation but not necessarily the entropy solution either. In Chapter 4 we perform some numerical experiments for the shallow water equations using filtered variables. Unfortunately, we have discovered that most of the tested methods are not suitable for the shallow water equations.

The questions whether the sequence of solutions $(u^\alpha)_{\alpha>0}$ converges to the entropy solution for continuous initial data and whether the limit u is a weak solution of Burgers' equation if arbitrary filters g are used, remain open.

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Chapter 2

Properties of the convectively filtered Burgers' equation

The aim of this chapter is to establish properties of the solutions to the convectively filtered Burgers' equation (CFB)

$$u_t^\alpha + \overline{u}^\alpha u_x^\alpha = 0, \quad (x,T) \in \mathbb{R}, \tag{2.1a}$$
$$\overline{u}^\alpha(x) = g_\alpha * u^\alpha(x), \quad x \in \mathbb{R}, \tag{2.1b}$$
$$u^\alpha(x,0) = u^0(x), \quad x \in \mathbb{R}, \tag{2.1c}$$

where $u^0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is the initial data, $g_\alpha(x) = \frac{1}{\alpha} g(\frac{x}{\alpha})$, and g is a nonnegative, symmetric and non-increasing (with respect to the absolute value of the argument) function with $\int g = 1$. In the following we will call a function g with the latter four properties a *filter*.

2.1 Well-Posedness and Convergence

Local existence and uniqueness of a solution to (2.1) has been shown in [2] using the Picard theorem on a Banach space (e.g. Theorem 4.1 in [16]), under the assumptions that the initial data $u^0 \in L^\infty(\mathbb{R})$ can be written as the sum of a bounded Lipschitz continuous function and an L^1-function, and that the filter g satisfies $g \in W^{2,1}(\mathbb{R})$ and $g'' \in L^\infty(\mathbb{R})$. If $u^0 \in BV(\mathbb{R})$, the same proof can be slightly modified to show local existence and uniqueness for $g \in W^{1,1}(\mathbb{R})$, which includes in particular the *Helmholtz filter*

$$g(x) = \frac{1}{2} e^{-|x|}. \tag{2.2}$$

If $u^0 \in C_b^1(\mathbb{R})$, we can even show local existence and uniqueness for (2.1) with g the *box filter*

$$g(x) = \frac{1}{2}\chi_{[-1,1]}(x). \tag{2.3}$$

In the same paper [2], global existence of the solution is proved, by showing that the characteristics of the equation,

$$\begin{aligned} \frac{d}{dt}\eta(X,t) &= \overline{u}^\alpha(\eta(X,t),t), \\ \eta(X,0) &= X, \end{aligned} \tag{2.4}$$

do not cross. This implies that the map η is a diffeomorphism for all times t and that if the initial condition is smooth, the solution of the equation stays smooth for all times. Moreover, since

$$\begin{aligned} \frac{d}{dt}u^\alpha(\eta(X,t),t) &= u_t^\alpha(\eta(X,t),t) + u_x^\alpha(\eta(X,t),t)\frac{d}{dt}\eta(X,t) \\ &= u_t^\alpha(\eta(X,t),t) + u_x^\alpha(\eta(X,t),t)\overline{u}^\alpha(\eta(X,t),t) \\ &= 0, \end{aligned}$$

by (2.4) and (2.1a), the solution can be expressed as a reparametrization of the initial data at any time t:

$$u(x,t) = u^0(\eta^{-1}(x,t)). \tag{2.5}$$

In particular, this implies that the L^∞-norm of the solution u^α is preserved. The same holds for the total variation: We let $\varphi \in C_0^1(\mathbb{R})$ with $|\varphi| \leq 1$. Using substitution in the integrals we compute

$$\begin{aligned} \int_{\mathbb{R}} u^\alpha(x,t)\varphi_x(x)\,dx &= \int_{\mathbb{R}} u^\alpha(\eta(X,t),t)\varphi_x(\eta(X,t))\eta_X(X,t)\,dX \\ &= \int_{\mathbb{R}} u^0(X)\frac{d}{dX}\varphi(\eta(X,t))\big(\eta_X(X,t)\big)^{-1}(\eta_X(X,t))\,dX \\ &= \int_{\mathbb{R}} u^0(X)\frac{d}{dX}\varphi(\eta(X,t))\,dX \\ &= \int_{\mathbb{R}} u^0(X)\frac{d}{dX}\widetilde{\varphi}(X)\,dX \end{aligned}$$

where we have denoted $\widetilde{\varphi} := \varphi \circ \eta$. Note that $|\widetilde{\varphi}(X)| \leq 1$ and $\widetilde{\varphi} \in C_0^1(\mathbb{R})$ since $\varphi \in C^1(\mathbb{R})$ and η is a diffeomorphism. Therefore, we can take the supremum over all $\varphi \in C_0^1(\mathbb{R})$ with $|\varphi| \leq 1$ in the above equation to obtain the claim by the definition of the total variation. Obviously, the maximum norm and the total variation are bounded independently of α by the maximum and the total variation of u^0. Easily, we find that the same bounds hold for the filtered velocity $\overline{u}^\alpha$, noting that

$$\begin{aligned} |\overline{u}^\alpha(x,t)| &= \left|\int_{\mathbb{R}} g_\alpha(y) u^\alpha(x-y,t)\,dy\right| \\ &\le \int_{\mathbb{R}} g_\alpha(y)|u^\alpha(x-y,t)|\,dy \\ &\le \|u^\alpha(\cdot,t)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} g_\alpha(y)\,dy \\ &= \|u^\alpha(\cdot,t)\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

where we used that g is normalized and nonnegative, and, applying Fubini's Theorem,

$$\begin{aligned} \mathrm{TV}(\overline{u}^\alpha(\cdot,t)) &= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int \overline{u}^\alpha(x,t)\varphi_x(x)\,dx \\ &= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int\int g(y)u^\alpha(x-\alpha y,t)\,dy\,\varphi_x(x)\,dx \\ &= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int g(y)\int u^\alpha(x-\alpha y,t)\varphi_x(x)\,dx\,dy \\ &\le \int g(y) \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int u^\alpha(x-\alpha y,t)\varphi_x(x)\,dx\,dy \\ &= \int g(y)\mathrm{TV}(u^\alpha(\cdot,t))\,dy \\ &= \mathrm{TV}(u^\alpha(\cdot,t)). \end{aligned}$$

Making use of the bounds on the L^∞-norm and the total variation, we can show in addition that u^α and $\overline{u}^\alpha$ are uniformly bounded in $C([0,T);L^1(\mathbb{R}))$.

To prove this, we first recall that for functions in $W^{1,1}(\mathbb{R})$, the total variation is equal to the L^1-norm of the derivative. Specifically, this implies for the total variation of the filtered velocity

$$\mathrm{TV}(\overline{u}^\alpha(\cdot,t)) = \int_{\mathbb{R}} |\overline{u}^\alpha_x(x,t)|\,dx.$$

Now we let $k > 0$, use that u^α satisfies (2.1a) and integrate by parts

$$\begin{aligned}
\int_{\mathbb{R}} |u^\alpha(x,t+k) - u^\alpha(x,t)|\,dx &= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int \varphi(x)(u^\alpha(x,t+k) - u^\alpha(x,t))\,dx \\
&= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int_{\mathbb{R}} \int_t^{t+k} \varphi(x) u_s^\alpha(x,s)\,ds\,dx \\
&= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int_{\mathbb{R}} \int_t^{t+k} -\varphi(x)\overline{u}^\alpha(x,s) u_x^\alpha(x,s)\,ds\,dx \\
&= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int_t^{t+k} \int_{\mathbb{R}} (\varphi(x)\overline{u}^\alpha(x,s))_x u^\alpha(x,s)\,dx\,ds \\
&\le \int_t^{t+k} \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int_{\mathbb{R}} (\varphi(x)\overline{u}^\alpha(x,s))_x u^\alpha(x,s)\,dx\,ds
\end{aligned} \tag{2.6}$$

We apply the product rule to the term $(\varphi(x)\overline{u}^\alpha(x,s))_x$, split the integral into the two summands which we consequently obtain, and estimate each of them separately:

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(x)\overline{u}_x^\alpha(x,s)u^\alpha(x,s)\,dx &\le \|\varphi\|_{L^\infty} \|u^\alpha(\cdot,s)\|_{L^\infty} \int_{\mathbb{R}} |\overline{u}_x^\alpha(x,s)|\,dx \\
&\le \|u^0\|_{L^\infty} \mathrm{TV}(u^0), \quad \forall \varphi \in C_0^1(\mathbb{R}), |\varphi| \le 1,
\end{aligned}$$

where we used the L^∞- and the total variation bounds on u^α and $\overline{u}^\alpha$. For the second term, we note that

$$\sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\le 1} \int_{\mathbb{R}} \varphi_x(x)\overline{u}^\alpha(x,s)u^\alpha(x,s)\,dx = \mathrm{TV}(\overline{u}^\alpha(\cdot,s)u^\alpha(\cdot,s)),$$

by definition. To see that this term is bounded, we need the equivalent definition of the total variation of a function

$$\mathrm{TV}(u) := \lim_{\varepsilon\to 0} \frac{1}{|\varepsilon|} \int |u(x+\varepsilon) - u(x)|\,dx$$

Thus, (omitting the time dependence of u^α and $\overline{u}^\alpha$).

$$\begin{aligned}
&\int |\overline{u}^\alpha(x+\varepsilon)u^\alpha(x+\varepsilon) - \overline{u}^\alpha(x)u^\alpha(x)|\,dx \\
&\leq \int |\overline{u}^\alpha(x+\varepsilon)u^\alpha(x+\varepsilon) - \overline{u}^\alpha(x)u^\alpha(x+\varepsilon)|\,dx \\
&\quad + \int |\overline{u}^\alpha(x)u^\alpha(x+\varepsilon) - \overline{u}^\alpha(x)u^\alpha(x)|\,dx \\
&\leq \|u^\alpha\|_{L^\infty} \int |\overline{u}^\alpha(x+\varepsilon) - \overline{u}^\alpha(x)|\,dx \\
&\quad + \|\overline{u}^\alpha\|_{L^\infty} \int |u^\alpha(x+\varepsilon) - u^\alpha(x)|\,dx \\
&\leq \|u^0\|_{L^\infty} \int |\overline{u}^\alpha(x+\varepsilon) - \overline{u}^\alpha(x)|\,dx \\
&\quad + \|u^0\|_{L^\infty} \int |u^\alpha(x+\varepsilon) - u^\alpha(x)|\,dx.
\end{aligned}$$

We divide the last term by $|\varepsilon|$ and take the limit $\varepsilon \to 0$ to obtain

$$\begin{aligned}
\mathrm{TV}(\overline{u}^\alpha(\cdot,s)u^\alpha(\cdot,s)) &\leq \|u^0\|_{L^\infty}\big(\mathrm{TV}(\overline{u}^\alpha(\cdot,s)) + \mathrm{TV}(u^\alpha(\cdot,s))\big) \\
&\leq 2\|u^0\|_{L^\infty}\mathrm{TV}(u^0)
\end{aligned}$$

Now we can bound (2.6) by

$$\int_{\mathbb{R}} |u^\alpha(x,t+k) - u^\alpha(x,t)|\,dx \leq 3\,\|u^0\|_{L^\infty}\mathrm{TV}(u^0)\,k,$$

which is the uniform bound we desired. The same bound follows for the filtered velocity by observing that

$$\|\overline{u}^\alpha(\cdot,t) - \overline{u}^\alpha(\cdot,t+k)\|_{L^1} \leq \|u^\alpha(\cdot,t) - u^\alpha(\cdot,t+k)\|_{L^1}.$$

In summary, we have,

Lemma 2.1 *[1] If $u^0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the solution u^α of (2.1) and the averaged quantity $\overline{u}^\alpha$ satisfy*

$$\|u^\alpha\|_{L^\infty}, \|\overline{u}^\alpha\|_{L^\infty} \leq \|u^0\|_{L^\infty}, \tag{2.7a}$$

$$TV(u^\alpha_x(\cdot,t))), TV(\overline{u}^\alpha_x(\cdot,t))) \leq TV(u^0), \tag{2.7b}$$

$$\begin{aligned}
\|u^\alpha(\cdot,t) - u^\alpha(\cdot,t+k)\|_{L^1} &\leq 3\|u^0\|_{L^\infty}TV(u^0)\,k, \quad k > 0, \\
\|\overline{u}^\alpha(\cdot,t) - \overline{u}^\alpha(\cdot,t+k)\|_{L^1} &\leq 3\|u^0\|_{L^\infty}TV(u^0)\,k, \quad k > 0.
\end{aligned} \tag{2.7c}$$

Lemma 2.1 together with Theorem A.8 in [12] (an application of Kolmogorov's Theorem) imply convergence of a subsequence u^{α_n} of solutions of (2.1) and of filtered velocities $\overline{u}^{\alpha_n}$ to a limit function u in $C([0,T);L^1_{\mathrm{loc}}(\mathbb{R}))$ Hence we have

Proposition 2.2 *[1] Suppose we solve (2.1) with initial data $u^0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, as $\alpha \to 0$, passing to a subsequence if necessary, there exists a function $u(x,t)$ with bounded total variation, such that*

$$u^\alpha, \overline{u}^\alpha \to u \text{ in } C([0,T); L^1_{loc}(\mathbb{R}))$$

That the limits of u^α and $\overline{u}^\alpha$ agree can be shown without effort, by noting that the convergence in $L^1_{\text{loc}}(\mathbb{R})$ implies convergence almost everywhere: For Ω bounded, we compute

$$\begin{aligned}\lim_{\alpha\to 0}\int_\Omega |u^\alpha(x,t) - \overline{u}^\alpha(x,t)|\,dx &= \lim_{\alpha\to 0}\int_\Omega \left| u^\alpha(x,t) - \int g(y)u^\alpha(x-\alpha y,t)\,dy\right| dx \\ &\le \lim_{\alpha\to 0}\int g(y)\int_\Omega |u^\alpha(x,t) - u^\alpha(x-\alpha y,t)|\,dx\,dy.\end{aligned}$$

We have used that g is nonnegative and normalized for the inequality. Now since Ω is bounded and $|u^\alpha(x,t) - u^\alpha(x-\alpha y,t)| \le 2\|u^0\|_{L^\infty}$, we can apply Lebesgue's Dominated Convergence Theorem and pass the limit under the integral signs.

Convergence to weak solutions of Burgers' equation

For filters whose Fourier transform can be written as

$$\hat{g}(k) = \frac{1}{1+\sum_{j=1}^n C_j k^{2j}}, \quad n < \infty,\ C_j \ge 0,\ C_n \ne 0, \tag{2.8}$$

it has been shown in addition that the limit function u in Proposition 2.2 is a weak solution of Burgers' equation [20]. Proving convergence to the entropy solution however seems to be quite involved. Reasons for this are given by the observations made in the following sections.

2.2 L^1-stability of the CFB equation with respect to initial data

Using some algebra on Equation (2.1a), we can prove the following L^1-stability estimate with respect to the initial data:

Proposition 2.3 *Let $u^0 \in W^{1,\infty}(\mathbb{R}) \cap BV(\mathbb{R})$ and $v^0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$. Denote $M_1 = \max\{\|u^0\|_\infty, \|v^0\|_\infty\}$, $M_2 = \|(u^0)'\|_\infty$ and $M_3 = \|(u^0)'\|_{L^1(\mathbb{R})}$. Let u, v be the solutions to (2.1a), (2.1b) with initial condition u^0, v^0 respectively and $g \in W^{1,1}(\mathbb{R})$ or the box filter (2.3). Then we have for $t \in (0,T)$ and a fixed $\alpha > 0$,*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \le e^{t\,(k_1\frac{M_1}{\alpha} + M_2 e^{t k_2 \frac{M_3}{\alpha}})}\|u^0 - v^0\|_{L^1(\mathbb{R})} \tag{2.9}$$

where k_1 and k_2 are constants depending on the filter g.

Proof Subtracting (2.1a) for v from the same equation for u and adding and subtracting the term $\overline{v}u_x$, we obtain

$$(u-v)_t + (\overline{u}-\overline{v})u_x + \overline{v}(u_x - v_x) = 0.$$

We multiply the equation by $\operatorname{sgn}(u-v)$, bring the second and third term to the right-hand side, and integrate over the spatial domain,

$$\int |u-v|_t \, dx = -\int \operatorname{sgn}(u-v)(\overline{u}-\overline{v})u_x \, dx - \int \overline{v}|u-v|_x \, dx.$$

Integrating the second term on the right-hand side by parts, we have

$$\begin{aligned}
\frac{d}{dt}\int |u-v| \, dx &= -\int \operatorname{sgn}(u-v)(\overline{u}-\overline{v})u_x \, dx + \int \overline{v}_x |u-v| \, dx \\
&\le \|u_x\|_\infty \int |\overline{u}-\overline{v}| \, dx + \|\overline{v}_x\|_\infty \int |u-v| \, dx \\
&= \|u_x\|_\infty \|\overline{u}-\overline{v}\|_{L^1(\mathbb{R})} + \|\overline{v}_x\|_\infty \, \|u-v\|_{L^1(\mathbb{R})} \\
&\le (\|u_x\|_\infty + \|\overline{v}_x\|_\infty) \, \|u-v\|_{L^1(\mathbb{R})} \qquad (2.10)
\end{aligned}$$

We estimate $\|\overline{v}_x\|_\infty$ and $\|u_x\|_\infty$: We start with the first one,

$$\overline{v}_x(x,t) = \int g'_\alpha(x-y)v(y,t)\, dy.$$

If $g \in W^{1,1}(\mathbb{R})$, this is well defined and we can estimate it by

$$\begin{aligned}
|\overline{v}_x(x,t)| &\le \frac{1}{\alpha}\|g'\|_{L^1(\mathbb{R})}\|v(\cdot,t)\|_\infty \\
&\le \frac{1}{\alpha}\|g'\|_{L^1(\mathbb{R})}\|v^0\|_\infty \\
&\le \frac{M_1}{\alpha}\|g'\|_{L^1(\mathbb{R})} \\
&\le \frac{k_1\, M_1}{\alpha} \qquad (2.11)
\end{aligned}$$

where we used that (2.1) preserves the maximum in the second inequality. If g is the box filter (2.3), its derivative does not exist. Since v is a function in $BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we can still estimate $\|\overline{v}_x\|_\infty$ as follows,

$$\begin{aligned}
|\overline{v}_x(x,t)| &= \left| \frac{d}{dx}\left(\frac{1}{2\alpha}\int_{x-\alpha}^{x+\alpha} v(y,t)\, dy \right) \right| \\
&= \left| \frac{1}{2\alpha}(v((x+\alpha)^-,t) - v((x-\alpha)^+,t)) \right| \\
&\le \frac{1}{\alpha}\|v(\cdot,t)\|_\infty \\
&\le \frac{\|v^0\|_\infty}{\alpha} \\
&\le \frac{M_1}{\alpha}
\end{aligned}$$

which is (2.11) for $k_1 = 1$. Again, we have made use of the fact that (2.1) preserves the maximum. We continue to estimate $\|u_x\|_\infty$. Firstly, we use the reparametrization of u in terms of the initial data (2.5) to estimate u_x:

$$\begin{aligned}\frac{d}{dx}u(x,t) &= \frac{d}{dx}(u^0(\eta^{-1}(x,t))) = (u^0)'(\eta^{-1}(x,t))\,\frac{d}{dx}\eta^{-1}(x,t)\\ &= (u^0)'(\eta^{-1}(x,t))\frac{1}{\eta_X(\eta^{-1}(x,t),t)}\end{aligned}$$

and therefore

$$\|u_x\|_\infty \le \|(u^0)'\|_\infty \sup_{X\in\mathbb{R}}\left|\frac{1}{\eta_X(X,t)}\right| \tag{2.12}$$

An estimate on $|\eta_X(X,t)|$ is derived in [2] for the purpose of proving global existence of the solution of (2.1). Specifically, in that paper, the ordinary differential equation (2.4) describing the evolution of the characteristics is rewritten as

$$\frac{d}{dt}\eta(X,t) = \int_{\mathbb{R}} g_\alpha(\eta(X,t)-\eta(Y,t))u^0(Y)\eta_Y(Y,t)dY. \tag{2.13}$$

Then they take the derivative with respect to X,

$$\frac{d}{dt}\eta_X(X,t) = \eta_X(X,t)\int_{\mathbb{R}} g'_\alpha(\eta(X,t)-\eta(Y,t))u^0(Y)\eta_Y(Y,t)dY,$$

divide the equation by $\eta_X(X,t)$ and integrate in t to obtain,

$$|\eta_X(X,t)| = \exp\left(\int_0^t\int_{\mathbb{R}} g_\alpha(\eta(X,s)-\eta(Y,s))(u^0)'(Y)\,dY\,ds\right)$$

We can bound this from above as follows:

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}} g_\alpha(\eta(X,s)-\eta(Y,s))(u^0)'(Y)\,dY\,ds\right|\\ &\qquad\qquad \le t\|g_\alpha\|_\infty\|(u^0)'\|_{L^1(\mathbb{R})} \le t\frac{k_2}{\alpha}\|(u^0)'\|_{L^1(\mathbb{R})},\end{aligned}$$

and therefore,

$$|\eta_X(X,t)| \ge \exp\Big(-t\frac{k_2}{\alpha}M_3\Big).$$

Combining this estimate with (2.12), we obtain

$$\|u_x\|_\infty \le \|(u^0)'\|_\infty \exp\Big(t\frac{k_2}{\alpha}M_3\Big). \tag{2.14}$$

Having estimated $\|u_x\|_\infty$ and $\|\overline{v}_x\|_\infty$ by (2.11) and (2.14), we can apply Gronwall's inequality in (2.10) to obtain the result. □

Obviously, the estimate of Theorem 2.3 is of no use in the limit $\alpha \to 0$. One rather wishes to have an estimate of the form

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \leq C \|u^0 - v^0\|_{L^1(\mathbb{R})}, \tag{2.15}$$

where C is a constant not depending on α. Unfortunately, this is not possible, as the following counterexample shows. For $K \geq 3$, we let

$$v^0(x) = \begin{cases} 0, & x < 0, \\ 1, & x \in [0, K], \\ 0, & x > K, \end{cases} \tag{2.16}$$

and

$$u^{0,\delta}(x) = \begin{cases} 0, & x < 0, \\ 0.5, & x \in [0, \delta), \\ 1, & x \in [\delta, K], \\ 0, & x > K, \end{cases} \tag{2.17}$$

for some $\delta > 0$. We have $\|v^0 - u^{0,\delta}\|_{L^1(\mathbb{R})} = \delta/2$ and thus $u^{0,\delta} \to v^0$ as $\delta \to 0$

Figure 2.1: Initial data v^0, (2.16) and $u^{0,\delta}$, (2.17).

in $L^1(\mathbb{R})$. We calculate the solutions to (2.1a), (2.1b) augmented with (2.16), (2.17) respectively. Since we know that the characteristics of the equation do not cross, we only need to find the speeds of the discontinuities, then we know that the solutions are constant inbetween. We will denote the discontinuities by s_1 and s_2 (from the left to the right) for the first problem and by s_1, s_2 and s_3 for the second problem. In the first problem, we have

$$\frac{d}{dt}s_1(t) = \int_0^{(s_2-s_1)(t)} g_\alpha(y)\,dy, \qquad s_1(0) = 0,$$
$$\frac{d}{dt}s_2(t) = \int_{(s_1-s_2)(t)}^{0} g_\alpha(y)\,dy, \qquad s_2(0) = K,$$

and therefore

$$\frac{d}{dt}(s_2 - s_1)(t) = 0, \quad (s_2 - s_1)(0) = K.$$

Thus $(s_2 - s_1)(t) = K$ for all t and the discontinuities satisfy actually

$$\frac{d}{dt} s_i(t) = \int_0^{K/\alpha} g(y)\, dy := G(\alpha), \quad i = 1, 2. \tag{2.18}$$

Hence the solution of (2.1a), (2.1b) and (2.16) is

$$v^\alpha(x,t) = \begin{cases} 0, & x < G(\alpha)t, \\ 1, & G(\alpha)t < x < G(\alpha)t + K, \\ 0, & x > G(\alpha)t + K, \end{cases} \tag{2.19}$$

(see Figure 2.2 for a plot of the characteristics of v^α).

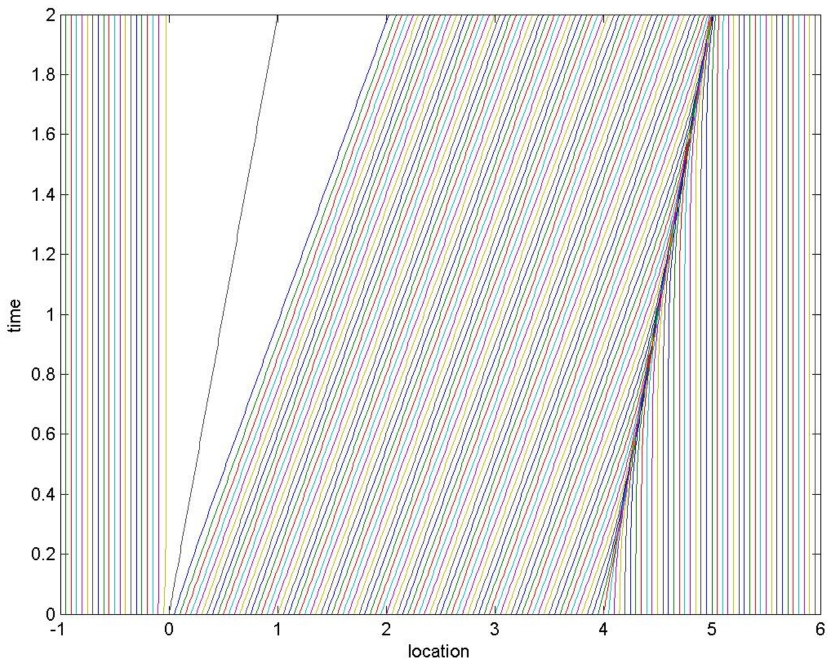

Figure 2.2: Characteristics for v^α, $\alpha = 0.05$, $K = 4$.

In the limit $\alpha \to 0$, this becomes

$$v(x,t) = \begin{cases} 0, & x < t/2, \\ 1, & t/2 < x < t/2 + K, \\ 0, & x > t/2 + K, \end{cases}$$

For the problem with initial data (2.17), the solution cannot be calculated so easily. We rather provide some estimates which suffice our purposes. We assume from now on $\delta < 1$ and $\alpha < 1$. For the discontinuities, we have,

$$\frac{d}{dt}s_1(t) = \frac{1}{2}\int_0^{(s_2-s_1)(t)} g_\alpha(y)\,dy + \int_{(s_2-s_1)(t)}^{(s_3-s_1)(t)} g_\alpha(y)\,dy, \qquad s_1(0)=0,$$

$$\frac{d}{dt}s_2(t) = \frac{1}{2}\int_{(s_1-s_2)(t)}^{0} g_\alpha(y)\,dy + \int_0^{(s_3-s_2)(t)} g_\alpha(y)\,dy, \qquad s_2(0)=\delta,$$

$$\frac{d}{dt}s_3(t) = \frac{1}{2}\int_{(s_1-s_3)(t)}^{(s_2-s_3)(t)} g_\alpha(y)\,dy + \int_{(s_2-s_3)(t)}^{0} g_\alpha(y)\,dy, \qquad s_3(0)=K,$$

The distances between the discontinuities, we denote them by $d_l := s_2 - s_1$, $d_r := s_3 - s_2$ and $d_m = s_3 - s_1$, satisfy

$$\frac{d}{dt}d_l(t) = \int_0^{(s_3-s_2)(t)} g_\alpha(y)\,dy - \int_{(s_2-s_1)(t)}^{(s_3-s_1)(t)} g_\alpha(y)\,dy \geq 0, \tag{2.20a}$$

$$\frac{d}{dt}d_r(t) = \frac{1}{2}\int_{(s_3-s_1)(t)}^{(s_2-s_3)(t)} g_\alpha(y)\,dy - \frac{1}{2}\int_{(s_1-s_2)(t)}^{0} g_\alpha(y)\,dy \geq -\frac{1}{4}, \tag{2.20b}$$

$$\begin{aligned}\frac{d}{dt}d_m(t) = {}& \frac{1}{2}\int_{(s_1-s_3)(t)}^{(s_2-s_3)(t)} g_\alpha(y)\,dy + \int_{(s_2-s_3)(t)}^{0} g_\alpha(y)\,dy \\ & - \frac{1}{2}\int_0^{(s_2-s_1)(t)} g_\alpha(y)\,dy + \int_{(s_2-s_1)(t)}^{(s_3-s_1)(t)} g_\alpha(y)\,dy \geq 0,\end{aligned} \tag{2.20c}$$

(we have used that g is a non-increasing function for the inequalities). So for $t < 4(K-\delta)$ the discontinuities will not meet, independently of α . For d_l we have in addition $\frac{d}{dt}d_l \leq 1/2$. We make this a bit more precise (omitting the variable t):

$$\begin{aligned}\frac{d}{dt}d_l &= \int_0^{s_2-s_1} g_\alpha(y)\,dy - \int_{s_3-s_2}^{s_3-s_1} g_\alpha(y)\,dy \\ &= \int_0^{(s_2-s_1)/\alpha} g(y)\,dy - \int_{(s_3-s_2)/\alpha}^{(s_3-s_1)/\alpha} g(y)\,dy \\ &= \int_0^{d_l/\alpha} g(y)\,dy - \int_{d_r/\alpha}^{(d_r+d_l)/\alpha} g(y)\,dy \\ &\geq \int_0^{d_l/\alpha} g(y)\,dy - \int_{(K-\delta-t/4)/\alpha}^{(K-\delta-t/4+d_l)/\alpha} g(y)\,dy,\end{aligned} \tag{2.21}$$

where we have used the bound (2.20b) in the last inequality. We assume now that $t \leq 4/3(K-2)$. This implies that $d_l \leq K-\delta-t/4$ and that we can estimate (2.21) further by

$$\begin{aligned}\frac{d}{dt}d_l &\geq \int_0^{d_l/\alpha} g(y)\,dy - \int_{d_l/\alpha}^{(K-t/4-\delta+d_l)/\alpha} g(y)\,dy, \\ &\geq \int_0^{d_l/\alpha} g(y)\,dy - \int_{d_l/\alpha}^{\infty} g(y)\,dy, \\ &\geq \int_0^{\delta/\alpha} g(y)\,dy - \int_{\delta/\alpha}^{\infty} g(y)\,dy,\end{aligned} \tag{2.22}$$

Now we fix $\delta > 0$ and choose $\alpha(\delta)$ so small that the last expression in (2.22) becomes $\geq 1/8$ for all $\alpha \leq \alpha(\delta)$. This is possible, since g is non-increasing and integrable. Then d_l satisfies

$$d_l(t) \geq \delta + t/8 \geq t/8, \quad \forall\, \alpha \leq \alpha(\delta). \tag{2.23}$$

We have chosen t small enough such that the discontinuities do not interact. Thus, the solution u^α of the convectively filtered Burgers' equation with initial data (2.17) has for all $0 < \alpha < 1$ and $t \leq 4/3(K-2)$ the form

$$u^{\alpha,\delta}(x,t) = \begin{cases} 0, & x < s_1^\alpha(t), \\ 0.5, & s_1^\alpha(t) < x < s_2^\alpha(t), \\ 1, & s_2^\alpha(t) < x < s_3^\alpha(t), \\ 0, & x > s_3^\alpha(t), \end{cases}$$

where $s_1^\alpha(t) + 2\delta \leq s_2^\alpha(t) + \delta \leq s_3^\alpha(t)$ uniformly in α (see Figure 2.3 for a plot of the characteristics of $u^{\alpha,\delta}$). So the limit function u^δ as $\alpha \to 0$, has the form

$$u^{\delta}(x,t) = \begin{cases} 0, & x < s_1^0(t), \\ 0.5, & s_1^0(t) < x < s_2^0(t), \\ 1, & s_2^0(t) < x < s_3^0(t), \\ 0, & x > s_3^0(t), \end{cases}$$

for some $s_1^0(t) < s_2^0(t) < s_3^0(t)$. In addition, we know from (2.23) that $s_2^0(t) - s_1^0(t) > t/8$. In this region, u^δ takes the value 0.5 whereas v is either 0 or 1.

Consequently, their difference in the L^1-norm satisfies

$$\|u^\delta(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \geq \frac{t}{16}, \tag{2.24}$$

while

$$\|u^{0,\delta} - v^0\|_{L^1(\mathbb{R})} \leq \frac{\delta}{2}. \tag{2.25}$$

The estimates (2.24), (2.25) are valid for arbitrary small $\delta > 0$. This shows that an estimate of the form (2.15) cannot be achieved for the solutions to the convectively filtered Burgers' equation, since it would imply the same bound for the limit functions obtained when letting $\alpha \to 0$. This cannot hold owing to (2.24) and (2.25).

An open question remains, whether there exists a bound on the difference of the solutions in L^1 of the form

$$\|u^\alpha(\cdot,t) - v^\alpha(\cdot,t)\|_{L^1(\mathbb{R})} \leq C\,\|u^0 - v^0\|_{L^\infty(\mathbb{R})}, \tag{2.26}$$

where C is a constant not depending on α. This could be helpful in proving convergence to the entropy solution for initial data in $C^0(\mathbb{R})$.

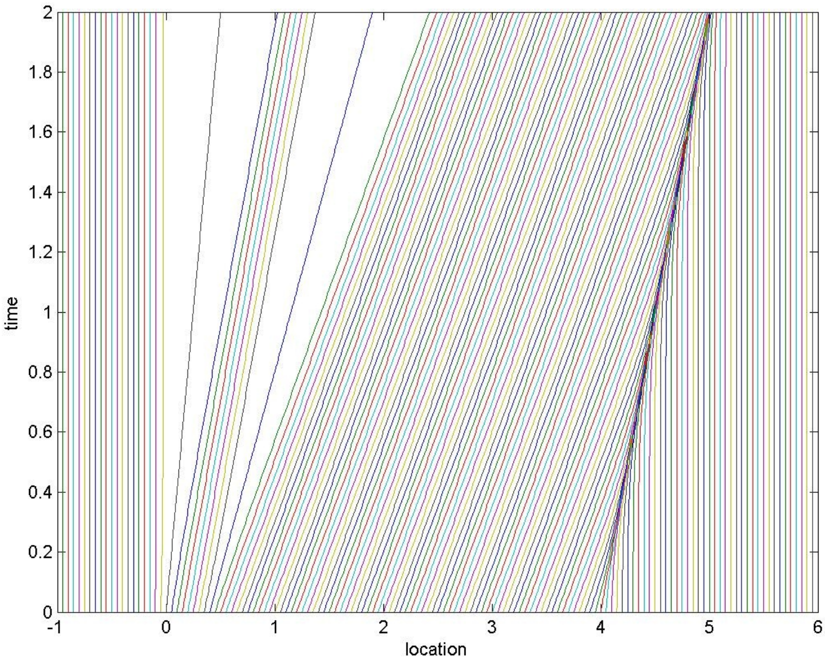

Figure 2.3: Characteristics for $u^{\alpha,\delta}$, $\alpha = 0.05$, $\delta = 0.4$, $K = 4$.

2.3 Reversibility

We will show now that the limit u of solutions u^α to the convectively filtered Burgers' equation (2.1) is not necessarily an entropy solution to the inviscid Burgers' equation due to the time reversibility of the regularized equation.

Proposition 2.4 *The convectively filtered Burgers' equation (2.1) does not imply an entropy inequality*

$$\eta(u)_t + q(u)_x \leq 0,$$

where η is a convex function and q with $q'(u) = u\eta'(u)$ its corresponding entropy flux, for the limit u of solutions u^α to (2.1).

Proof We will prove the proposition by a contradiction argument.

Let u^α denote the solution of the CFB equation (2.1) and $v^\alpha(x,t) := u^\alpha(-x,-t)$, $(x,t) \in \mathbb{R} \times (0,T)$ its reflection in time and space. Observing that

$$\overline{v}^\alpha(x,t) = \int_{\mathbb{R}} u^\alpha(-x-y,-t) g_\alpha(y)\, dy = \overline{u}^\alpha(-x,-t)$$

and

$$\frac{d}{dt}v^\alpha(x,t) = -u_t^\alpha(-x,-t), \quad \frac{d}{dx}v^\alpha(x,t) = -u_x^\alpha(-x,-t),$$

we obtain that v^α satisfies the equation

$$-v_t^\alpha(x,t) - \overline{v}^\alpha(x,t)v_x^\alpha(x,t) = 0 \quad (x,t) \in \mathbb{R} \times (0,T)$$

which is the same differential equation as the one u^α satisfies (except for the initial data which is reflected). If we assume that the limit function u obtained when letting $\alpha \to 0$, is a weak solution to Burgers' equation,

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{2.27a}$$

$$u(x,0) = u^0(x), \quad x \in \mathbb{R}, \tag{2.27b}$$

then the limit v of the sequence of functions $v^\alpha(x,t) = u^\alpha(-x,-t)$ obtained as $\alpha \to 0$ will also be a weak solution of Burgers' equation with reflected initial data since it satisfies the same equation. If we furthermore assume that (2.1a) implies that its solution u^α satisfies an entropy inequality, then, since v^α is a solution of the same equation, this should imply that v^α satisfies the same entropy inequality. This is a contradiction, which can easily be seen by the following argument:

We consider the viscous approximation to Burgers' equation,

$$u_t^\varepsilon + \left(\frac{u^{\varepsilon\,2}}{2}\right)_x = \varepsilon\, u_{xx}^\varepsilon, \quad (x,t) \in \mathbb{R} \times (0,T),$$
$$u^\varepsilon(x,0) = u^0(x), \quad x \in \mathbb{R}.$$

We know that the solution u^ε of this equation converges, as $\varepsilon \to 0$, to a weak solution u of Burgers' equation which satisfies in addition the entropy inequality

$$\eta(u)_t + q(u)_x \leq 0,$$

for any convex function η and corresponding entropy flux q with $q'(u) = u\eta'(u)$.

If we again consider the reversion of u^ε in time and space, $v^\varepsilon(x,t) = u^\varepsilon(-x,-t)$, we notice that v^ε is a solution of the equation

$$v_t^\varepsilon + \left(\frac{v^{\varepsilon\,2}}{2}\right)_x = -\varepsilon\, v_{xx}^\varepsilon, \quad (x,t) \in \mathbb{R} \times (0,T),$$
$$v^\varepsilon(x,0) = u^0(-x), \quad x \in \mathbb{R}.$$

This implies that the limit function v obtained from $(v^\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$, if it exists, satisfies the opposite entropy inequality than the limit u of the functions u^ε, namely

$$\eta(v)_t + q(v)_x \geq 0,$$

where η denotes the entropy and q the entropy flux. We know that the entropy inequality satisfied by the entropy solution can be strict, as it is the case for example if shocks occur in the solution. Then, the limit functions u and v satisfy different inequalities which contradicts the previous reasoning. Thus, the convectively filtered Burgers' equation does not imply an entropy inequality and the limit $u^\alpha \to u$ is not necessarily an entropy solution. □

An explicit example of an initial data for (2.1) for which the solution u^α does not converge to the entropy solution of Burgers' equation has been given in the paper of Bhat and Fetecau [2]. They calculate the solution to (2.1) for a Riemann Problem with $u_L < u_R$ and show that in the limit $\alpha \to 0$, it converges to a weak solution with a non-entropic shock wave.

Nevertheless it has been conjectured, and numerical experiments show some evidence it might be true ([1] and my own experiments), that for continuous initial data, u^α converges to the entropy solution of Burgers' equation, at least if we use the Helmholtz filter (2.2) for g. However, I find it very difficult to show this and have been unable to do it so far. The lack of a uniform L^1-stability in α with respect to the initial data seems to make it additionally hard, since this makes it almost impossible to find an independently of α convergent numerical scheme.

2.4 Filtering for conservation laws with general fluxes

In spite of the absence of a proof that the solution of (2.1) converges for continuous initial data to the entropy solution of Burgers' equation as $\alpha \to 0$, it has at least been shown that it converges to a weak solution if filters which have a Fourier transform given by (2.8) are employed [20], as we have mentioned in the first section of this chapter. Nevertheless, since the convectively filtered Burgers' equation has mainly been analyzed with the aim of applying averaging of variables to systems of conservation laws such as the Euler equations, it might be interesting to examine firstly the case of a general scalar conservation law

$$u_t + f(u)_x = 0, \tag{2.28}$$

with a $C^2(\mathbb{R})$-flux function f. There are at least two possible ways how one could mimic the regularization in (2.1) for the more general conservation law (2.28). One one hand, we can filter the derivative of the flux function

$$u_t^\alpha + \overline{f'(u^\alpha)} u_x^\alpha = 0, \tag{2.29a}$$

$$\overline{f'(u^\alpha(x))} = \big(g_\alpha * f'(u^\alpha)\big)(x), \tag{2.29b}$$

$$u^\alpha(x,0) = u^0(x), \tag{2.29c}$$

on the other hand, we can filter u^α in the flux function:

$$u_t^\alpha + f'(\overline{u}^\alpha) u_x^\alpha = 0, \tag{2.30a}$$

$$\overline{u}^\alpha(x) = \big(g_\alpha * u^\alpha\big)(x), \tag{2.30b}$$

$$u^\alpha(x,0) = u^0(x), \tag{2.30c}$$

Local and global existence can be shown using the method of characteristics, in an analogous manner as it is done for the convectively filtered Burgers' equation. Hence we can write the solution u^α at time t as a reparametrization of the initial data u^0 which implies that the maximum and the total variation of the solution are conserved (see Section 2.1). Likewise, we can show Lipschitz continuity in time for u^α. Applying then Kolmogorov's Theorem (e.g. Theorem A.8 in [12] for the particular case we are considering here), we obtain convergence of a subsequence u^{α_n} of solutions to (2.29), (2.30) respectively, to a function u of bounded variation in $C([0,T); L^1_{\mathrm{loc}}(\mathbb{R}))$. However, u is not necessarily a weak solution of inviscid Burgers' equation, as the following counterexample illustrates. Specifically, we consider a Riemann problem with initial data

$$u^0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \tag{2.31}$$

We compute the solution to this problem at time t using the characteristics, similarly to how it is done in [2]. For this purpose, we define

$$h^-(x) = \int_{-\infty}^x g(y)\, dy, \quad x < 0; \quad h^+(x) = -\int_x^\infty g(y)\, dy, \quad x > 0. \tag{2.32}$$

We start by calculating the solution of the modification (2.29a), (2.29b) with the Riemann initial data (2.31). Then the characteristics η satisfy the ordinary differential equation (compare with (2.13))

$$\begin{aligned} \frac{d}{dt}\eta(X,t) = {} & \frac{f'(u_L)}{\alpha} \int_{-\infty}^0 g\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right) \eta_Y(Y,t)\, dY \\ & + \frac{f'(u_R)}{\alpha} \int_0^\infty g\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right) \eta_Y(Y,t)\, dY \end{aligned} \tag{2.33}$$

For $X < 0$, this becomes

$$\begin{aligned}
\frac{d}{dt}\eta(X,t) &= \frac{f'(u_L)}{\alpha}\int_{-\infty}^{X} g\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\
&+ \frac{f'(u_L)}{\alpha}\int_{X}^{0} g\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\
&+ \frac{f'(u_R)}{\alpha}\int_{0}^{\infty} g\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\
&= -f'(u_L)\left[h^{+}\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{-\infty}^{X} \\
&- f'(u_L)\left[h^{-}\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{X}^{0} \\
&- f'(u_R)\left[h^{-}\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{0}^{\infty}
\end{aligned}$$

Since $\eta(Y,t) \to \pm\infty$ as $Y \to \pm\infty$ and $h^{\pm}$ decay at infinity, we can simplify the last expression to obtain

$$\frac{d}{dt}\eta(X,t) = f'(u_L) + (f'(u_R) - f'(u_L))h^{-}\left(\frac{\eta(X,t)-\eta(0,t)}{\alpha}\right). \tag{2.34}$$

If $X = 0$, (2.33) becomes

$$\begin{aligned}
\frac{d}{dt}\eta(0,t) &= \frac{f'(u_L)}{\alpha}\int_{-\infty}^{0} g\left(\frac{\eta(0,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\
&+ \frac{f'(u_R)}{\alpha}\int_{0}^{\infty} g\left(\frac{\eta(0,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\
&= -f'(u_L)\left[h^{+}\left(\frac{\eta(0,t)-\eta(Y,t)}{\alpha}\right)\right]_{-\infty}^{0} \\
&- f'(u_R)\left[h^{-}\left(\frac{\eta(0,t)-\eta(Y,t)}{\alpha}\right)\right]_{0}^{\infty}.
\end{aligned}$$

Again, using $\eta(Y,t) \to \pm\infty$ as $Y \to \pm\infty$ and the decay of $h^{\pm}$ at infinity, we get

$$\frac{d}{dt}\eta(0,t) = \frac{1}{2}(f'(u_L) + f'(u_R)). \tag{2.35}$$

In the case $X > 0$, we have

$$
\begin{aligned}
\frac{d}{dt}\eta(X,t) &= \frac{f'(u_L)}{\alpha}\int_{-\infty}^{0} g\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\
&+ \frac{f'(u_R)}{\alpha}\int_{0}^{X} g\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\
&+ \frac{f'(u_R)}{\alpha}\int_{X}^{\infty} g\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\eta_Y(Y,t)\,dY \\
&= -f'(u_L)\left[h^+\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{-\infty}^{0} \\
&- f'(u_R)\left[h^+\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{0}^{X} \\
&- f'(u_R)\left[h^-\left(\frac{\eta(X,t)-\eta(Y,t)}{\alpha}\right)\right]_{X}^{\infty}
\end{aligned}
$$

yielding

$$
\frac{d}{dt}\eta(X,t) = f'(u_R) + (f'(u_R) - f'(u_L))h^+\left(\frac{\eta(X,t)-\eta(0,t)}{\alpha}\right). \tag{2.36}
$$

Solving the ordinary differential equation (2.35), we obtain

$$
\eta(0,t) = \frac{1}{2}(f'(u_L) + f'(u_R))t. \tag{2.37}
$$

Now we can plug in the value of $\eta(0,t)$ in (2.34) and (2.36) and solve the differential equations. But knowing that the characteristics do not cross and that u^α is constant along the characteristics, this is not necessary to find u^α. If $x < (f'(u_L) + f'(u_R))t/2$, it lies on a characteristic emerging from some $X < 0$ and therefore $u^\alpha(x,t) = u_L$, on the other hand, if $x > (f'(u_L) + f'(u_R))t/2$, it lies on a characteristic emerging from some $X > 0$ and therefore $u^\alpha(x,t) = u_R$. So the solution to (2.29a), (2.29b) with the Riemann initial data (2.31) is given by

$$
u^\alpha(x,t) = \begin{cases} u_L, & x < (f'(u_L) + f'(u_R))t/2, \\ u_R, & x > (f'(u_L) + f'(u_R))t/2, \end{cases} \tag{2.38}
$$

We notice that this solution is independent of the parameter α and the limit $\alpha \to 0$ is trivial. But (2.38) is not a weak solution of Burgers' equation because the speed of the discontinuity does not satisfy the Rankine-Hugoniot condition, except if

$$
\frac{f'(u_L) + f'(u_R)}{2} = \frac{f(u_L) - f(u_R)}{u_L - u_R},
$$

which is in general not the case.

In order to solve (2.30a), (2.30b) with the Riemann initial data (2.31), we proceed similarly (in fact only a small modification in the calculations in [2] is needed), to obtain the solution

$$u^\alpha(x,t) = \begin{cases} u_L, & x < f'((u_L + u_R)/2)t, \\ u_R, & x > f'((u_L + u_R)/2)t. \end{cases} \tag{2.39}$$

We notice as before that the speed of the discontinuity does not satisfy the Rankine-Hugoniot condition in general.

So we see that not even in the case of a scalar conservation law, arbitrarily applying averaging of variables leads to a satisfactory result. Thus in the case of a system of conservation laws, it might be even more difficult to find an appropriate regularization of the equation by means of filtering variables. Numerical experiments maintainig this conclusion are conducted in Chapter 4.

Chapter 3

A time dependent filter

We have seen in Chapter 2 that the convectively filtered Burgers' equation is reversible in time which excludes the existence of an entropy inequality for the limit u of solutions u^α to (2.1). For this reason, we will investigate the CFB equation with a filter depending on time in this chapter. Specifically, we will analyze the following equations which are not reversible in time:

$$u_t^\alpha + \overline{u}^\alpha u_x^\alpha = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{3.1a}$$
$$\overline{u}^\alpha + \alpha(\overline{u}_t^\alpha - \overline{u}_{xx}^\alpha) = u^\alpha, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{3.1b}$$
$$u^\alpha(x,0) = u^{0,\alpha}(x), \quad x \in \mathbb{R}, \tag{3.1c}$$
$$u^{0,\alpha}(x) = (\omega_\alpha * u^0)(x), \quad x \in \mathbb{R}, \tag{3.1d}$$

where we choose $u^0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and

$$\omega_\alpha(x) = \frac{1}{\alpha^K}\omega\left(\frac{x}{\alpha^K}\right), \quad x \in \mathbb{R}, \tag{3.2}$$

for a symmetric, nonnegative $\omega \in C_0^1(\mathbb{R})$ with $\int_{\mathbb{R}} \omega = 1$ and $K \in \mathbb{N}$. This causes the initial data $u^{0,\alpha}$ to be not only in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ but also in $C_b^1(\mathbb{R})$, of which we will make use in the local existence proof for the equations. Nevertheless, it seems that solutions to (3.1a), (3.1b) also exist for less regular initial data, for example, one can construct an explicit solution to (3.1a), (3.1b) for Riemann initial data using the method of characteristics. If we want $u^{0,\alpha}$ to converge faster to u^0 as $\alpha \to 0$, we can choose a larger exponent K.

In order to see that (3.1) is indeed not reversible in time, we set $v^\alpha(x,t) := u^\alpha(-x,-t)$ and $\overline{v}^\alpha(x,t) := \overline{u}^\alpha(-x,-t)$. Then v^α and $\overline{v}^\alpha$ satisfy the differential equations

$$v_t^\alpha + \overline{v}^\alpha v_x^\alpha = 0,$$
$$\overline{v}^\alpha - \alpha(\overline{v}_t^\alpha + \overline{v}_{xx}^\alpha) = v^\alpha,$$

which differ from (3.1a), (3.1b) in the second equation.

The aim of this chapter is to show well-posedness of (3.1), find estimates on the solution and investigate whether the solution u^α converges to the entropy solution of the inviscid Burgers' equation as $\alpha \to 0$. We choose the particular time dependent regularization (3.1b) since it has the advantage that we have an explicit expression for the Green function associated to the differential operator $\mathcal{A} := (\mathbb{I} + \alpha(\partial_t - \partial_{xx}^2))$, that is, the solution $\overline{u}^\alpha$ of the second equation (3.1b) can be written in terms of u^α as

$$\overline{u}^\alpha(x,t) = \int_0^t \int_{\mathbb{R}} e^{-\frac{t-s}{\alpha}} \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{-(x-y)^2}{4(t-s)}} \frac{u^\alpha(y,s)}{\alpha} \, dy \, ds. \tag{3.3}$$

We denote the associated Green function

$$\Phi_\alpha(x,t) = \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}. \tag{3.4}$$

We see that Φ_α is nonnegative, symmetric in the variable x and decreasing with respect to the absolute values of the arguments x and t. For differential operators with higher order derivatives in x or t, it is more difficult to find (if possible) the associated Green function and it does not necessarily have the latter mentioned nice properties. Furthermore, the same filter was found to be more suitable to regularize the phase mobility in a model for two-phase flow in porous media than the Helmholtz filter

$$g_\alpha(x) = \frac{1}{2\alpha} e^{-|x|/\alpha},$$

[7].

The Green function Φ_α

As already mentioned, Φ_α is nonnegative, symmetric with respect to the spatial variable and non-increasing with respect to t and $|x|$. We also note that Φ_α has a singularity at $(x,t) = (0,0)$. Nevertheless, we have bounds on the L^1-norm of Φ_α and $\Phi_{\alpha,x}$:

Lemma 3.1 *For all $0 < \tau \le T$, the time dependent filter (3.4) satisfies the bounds*

$$\|\Phi_\alpha\|_{L^1(\mathbb{R}\times(0,\tau))} = 1 - e^{-\frac{\tau}{\alpha}}, \tag{3.5}$$

and

$$\|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,\tau))} \le \frac{3}{\sqrt{\pi\alpha}}. \tag{3.6}$$

Proof The estimates follow by simple calculations. We start with (3.5):

$$
\begin{aligned}
\|\Phi_\alpha\|_{L^1(\mathbb{R}\times(0,\tau))} &= \int_0^\tau \int_{\mathbb{R}} \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \, dx \, dt \\
&= \int_0^\tau \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{x^2}{4t}} \, dx \, dt \\
&= \int_0^\tau \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-v^2} \, dv \, dt \\
&= \int_0^\tau \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \, dt \\
&= -e^{-\frac{t}{\alpha}} \Big|_{t=0}^{t=\tau} \\
&= 1 - e^{-\frac{\tau}{\alpha}}.
\end{aligned}
$$

For the second bound, we compute

$$
\begin{aligned}
\|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,\tau))} &= \int_0^\tau \int_{\mathbb{R}} \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{4\pi t}} \left| \frac{\partial}{\partial x} e^{\frac{-x^2}{4t}} \right| dx \, dt \\
&= \int_0^\tau \int_{\mathbb{R}} \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{4\pi t}} \left| \frac{x}{2t} \right| e^{\frac{-x^2}{4t}} \, dx \, dt \\
&= \int_0^\tau \int_0^\infty \frac{2}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{4\pi t}} \frac{x}{2t} e^{\frac{-x^2}{4t}} \, dx \, dt \\
&= \int_0^\tau \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{\pi t}} \int_0^\infty \frac{x}{2t} e^{\frac{-x^2}{4t}} \, dx \, dt \\
&= \int_0^\tau \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{\pi t}} \; -e^{\frac{-x^2}{4t}} \Big|_0^\infty \, dt \\
&= \int_0^\tau \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \frac{1}{\sqrt{\pi t}} \, dt \\
&= \int_0^{\tau/\alpha} \frac{1}{\sqrt{\pi \alpha s}} e^{-s} \, ds \\
&\leq \frac{1}{\sqrt{\pi\alpha}} \left\{ \int_0^1 \frac{1}{\sqrt{s}} \, ds + \int_1^\infty e^{-s} \, ds \right\} \\
&\leq \frac{3}{\sqrt{\pi\alpha}}.
\end{aligned}
$$

□

3.1 Existence and Uniqueness

3.1.1 Local existence and uniqueness

We proceed to showing local existence and uniqueness of the solution to (3.1). In order to simplify the notation we will omit explicitely writing the dependence of the solution of (3.1) on α in this paragraph. Similarly, we will write u^0 instead of $u^{0,\alpha}$ to denote the regularized initial data. To show local existence and uniqueness, we will proceed as in [2] with some modifications.

Characteristic equations

Firstly, we recall the characteristics equations for (3.1). These are, denoting the characteristics by η, similarly to Chapter 2,

$$\frac{d}{dt}\eta(X,t) = \overline{u}(\eta(X,t),t), \quad (X,t) \in \mathbb{R} \times (0,T), \tag{3.7a}$$

$$\eta(X,0) = X, \quad X \in \mathbb{R}. \tag{3.7b}$$

Then formally

$$\frac{d}{dt}u(\eta(X,t),t) = u_t(\eta(X,t),t) + u_x(\eta(X,t),t)\frac{d}{dt}\eta(X,t) = 0 \tag{3.8}$$

and we can therefore rewrite the solution at time t as a reparametrization of the initial data,

$$u(\eta(X,t),t) = u^0(X). \tag{3.9}$$

We use the definition of $\overline{u}$ to rewrite the first equation in (3.7) in the form

$$\frac{d}{dt}\eta(X,t) = \int_0^t \int_{\mathbb{R}} \Phi_\alpha(\eta(X,t) - y, t-s)\, u(y,s)\, dy\, ds. \tag{3.10}$$

Assuming that $\eta(X,t)$ is a diffeomorphism, we change variables $y = \eta(Y,s)$ to obtain

$$\begin{aligned}\frac{d}{dt}\eta(X,t) &= \int_0^t \int_{\mathbb{R}} \Phi_\alpha(\eta(X,t) - \eta(Y,s), t-s)\, u(\eta(Y,s),s)\, \eta_Y(Y,s)\, dY\, ds \\ &= \int_0^t \int_{\mathbb{R}} \Phi_\alpha(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, \eta_Y(Y,s)\, dY\, ds,\end{aligned} \tag{3.11}$$

where we have used (3.9). We differentiate (3.11) with respect to X and get

$$\frac{d}{dt}\eta_X(X,t) = \eta_X(X,t) \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, \eta_Y(Y,s)\, dY\, ds \tag{3.12}$$

Using integration by parts in the variable Y, this equation can also be expressed in the form

$$\frac{d}{dt}\eta_X(X,t) = \eta_X(X,t) \int_0^t \int_{\mathbb{R}} \Phi_\alpha(\eta(X,t) - \eta(Y,s), t-s)\, (u^0)'(Y)\, dY\, ds \tag{3.13}$$

Well-posedness in B

Now we substitute

$$f(X,t) = \eta_X(X,t) - 1. \tag{3.14}$$

Then we can rewrite the characteristic map as

$$\eta(X,t) = X + \int_{-\infty}^{X} f(Z,t)\,dZ. \tag{3.15}$$

Moreover, $f(X,t)$ solves

$$\frac{df}{dt}(X,t) = \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}\left(X + \int_{-\infty}^{X} f(Z,t)\,dZ - Y - \int_{-\infty}^{Y} f(Z,s)\,dZ, t-s\right) \cdot u^0(Y)\,(1+f(Y,s))\,dY\,ds \cdot (1+f(X,t)) \tag{3.16}$$

with initial condition

$$f(X,0) = 0. \tag{3.17}$$

We want to show that (3.16) is locally well posed in a suitable Banach space. To be precise, we take $[0,\delta)$, $\delta > 0$, to be a small time interval and we define the Banach space $\mathbf{B}$ to be the completion of the normed vector space of functions $f : \mathbb{R} \times [0,\delta) \to \mathbb{R}$ such that $\|f\|_{\mathbf{B}} < \infty$ where $\|\cdot\|_{\mathbf{B}}$ is defined by

$$\|f\|_{\mathbf{B}} = \sup_{t\in[0,\delta)} \|f(t)\|_{L^\infty(\mathbb{R})} + \sup_{X\in\mathbb{R}, t\in[0,\delta)} \left|\int_{-\infty}^{X} f(Z,t)dZ\right|. \tag{3.18}$$

We fix a parameter $\gamma \in (0,1)$ and define $U \in \mathbf{B}$ to be the open set given by all $f \in \mathbf{B}$ for which $\|f\|_{\mathbf{B}} < 1-\gamma$. We define the functional $F(f)$ by the right-hand side of the equation in (3.16):

$$F(f) = \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}\left(X + \int_{-\infty}^{X} f(Z,t)dZ - Y - \int_{-\infty}^{Y} f(Z,s)dZ, t-s\right) \cdot u^0(Y)\,(1+f(Y,s))\,dY\,ds \cdot (1+f(X,t)) \tag{3.19}$$

Proposition 3.2 *There exists $\varepsilon > 0$ and a unique C^1-integral curve $f : [0,\varepsilon) \to U$, $t \mapsto f(t)$ satisfying the initial-value problem*

$$\frac{df}{dt} = F(f(t)), \quad f(\cdot,0) = 0 \tag{3.20}$$

Proof If we can show that F maps U to $\mathbf{B}$ and that F is Lipschitz on U with respect to the $\mathbf{B}$-norm, i.e. that there exists $c > 0$ such that for all $f, g \in U$

$$\|F(f) - F(g)\|_{\mathbf{B}} \leq c\,\|f - g\|_{\mathbf{B}}$$

then we can apply the Picard theorem on a Banach space (see e.g. Theorem 4.1 in [16]) to conclude the existence of a unique solution to (3.20) on a short time interval.
If we can show the Lipschitz continuity of F, the fact that $F : U \to \mathbf{B}$ follows since

$$\|F(f)\|_{\mathbf{B}} = \|F(f) - F(0)\|_{\mathbf{B}} + \|F(0)\|_{\mathbf{B}} \leq c\,\|f\|_{\mathbf{B}} + \|F(0)\|_{\mathbf{B}} \tag{3.21}$$

and

$$\begin{aligned}
\|F(0)(t)\|_\infty &= \left\| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(X - Y, t - s)\, u^0(Y)\, dY\, ds \right\|_\infty \\
&\leq \left\| \int_0^t \int_{\mathbb{R}} |\Phi_{\alpha,x}(X - Y, t - s)|\, dY\, ds \right\|_\infty \|u^0\|_\infty \\
&\leq \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))} \|u^0\|_\infty,
\end{aligned}$$

$$\begin{aligned}
\left| \int_{-\infty}^X F(0)(t)(Z) dZ \right| &= \left| \int_{-\infty}^X \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(Z - Y, t - s)\, u^0(Y)\, dY\, ds\, dZ \right| \\
&\leq \int_{-\infty}^X \int_0^t \int_{\mathbb{R}} |\Phi_{\alpha,x}(Y, t - s)|\, |u^0(Z - Y)|\, dY\, ds\, dZ \\
&= \int_0^t \int_{\mathbb{R}} |\Phi_{\alpha,x}(Y, t - s)| \int_{-\infty}^X |u^0(Z - Y)|\, dZ\, dY\, ds \\
&\leq \|u^0\|_{L^1(\mathbb{R})} \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))}
\end{aligned}$$

where we have made use of the Tonelli Theorem in the second last step. Note that both of the above estimates are independent of the time t and hence, by (3.6), $\|F(0)\|_{\mathbf{B}} \leq C$. Therefore the right-hand side in (3.21) is finite.

Proof of the Lipschitz continuity:
Let $f, g \in U$. We denote

$$\eta(X, t) = X + \int_{-\infty}^X f(Z, t)\, dZ \tag{3.22}$$

and

$$\xi(X, t) = X + \int_{-\infty}^X g(Z, t)\, dZ \tag{3.23}$$

which is well defined since $f, g \in \mathbf{B}$. We start by estimating $\sup_{t\in[0,\delta)} \|F(f)(t) - F(g)(t)\|_\infty$. We rewrite the difference $F(f) - F(g)$ as

$$\begin{aligned}
&F(f) - F(g) = \\
&(f(X,t) - g(X,t)) \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, (1 + f(Y,s))\, dY\, ds \\
&+ (1 + g(X,t)) \Big[\int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, (1 + f(Y,s))\, dY\, ds \\
&\qquad - \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t) - \xi(Y,s), t-s)\, u^0(Y)\, (1 + g(Y,s))\, dY\, ds \Big]. \qquad (3.24)
\end{aligned}$$

The first term on the right-hand side of (3.24) can be estimated as follows

$$\begin{aligned}
&\left\| (f(X,t) - g(X,t)) \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, (1 + f(Y,s))\, dY\, ds \right\|_\infty \\
&\leq \|f(t) - g(t)\|_\infty \left\| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, (1 + f(Y,s))\, dY\, ds \right\|_\infty .
\end{aligned}$$

Since $f \in \mathbf{B}$, we have $\|f(t)\|_\infty < 1 - \gamma$ for $t \in [0,\delta)$ and therefore it holds $1 + f(Z,t) > \gamma > 0$ almost everywhere. This implies by (3.22) that $\eta(\cdot,t)$ is a monotone increasing differentiable function with range $\mathbb{R}$, so in fact a diffeomorphism of $\mathbb{R}$ for every fixed t small enough. Hence we have

$$\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, (1 + f(Y,s))\, dY\, ds \right\|_\infty \\
&\quad = \left\| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, \eta_Y(Y,s)\, dY\, ds \right\|_\infty \\
&\quad = \left\| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - y, t-s)\, u^0(\eta^{-1}(y,s))\, dy\, ds \right\|_\infty \\
&\quad \leq \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))} \|u^0\|_\infty
\end{aligned}$$

where we changed variables $y = \eta(Y,s)$ in the second step. $\|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))}$ is bounded by (3.6), so the first term in (3.24) is bounded:

$$\begin{aligned}
&\left\| (f(X,t) - g(X,t)) \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, (1 + f(Y,s))\, dY\, ds \right\|_\infty \\
&\qquad\qquad \leq \|f(t) - g(t)\|_\infty \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))} \|u^0\|_\infty
\end{aligned}$$

The terms $\|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))}$ and $\|u^0\|_\infty$ are independent of t, so we can take the supremum over all $t \in [0,\delta)$ to obtain:

$$\begin{aligned}
&\sup_{t\in[0,\delta)} \left\| (f(X,t) - g(X,t)) \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, (1 + f(Y,s))\, dY\, ds \right\|_\infty \\
&\qquad\qquad \leq \|f - g\|_{\mathbf{B}} \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))} \|u^0\|_\infty . \qquad (3.25)
\end{aligned}$$

Let us pass to the second term on the right-hand side of (3.24).

$$\begin{aligned}
&\Big\|(1+g(X,t))\Big[\int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t)-\eta(Y,s),t-s)\, u^0(Y)\,(1+f(Y,s))\,dY\,ds \\
&\qquad - \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t)-\xi(Y,s),t-s)\, u^0(Y)\,(1+g(Y,s))\,dY\,ds\Big]\Big\|_\infty \\
&\leq \|1+g(t)\|_\infty \Big\|\int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t)-\eta(Y,s),t-s)\, u^0(Y)\,(1+f(Y,s))\,dY\,ds \\
&\qquad - \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t)-\xi(Y,s),t-s)\, u^0(Y)\,(1+g(Y,s))\,dY\,ds\Big\|_\infty .
\end{aligned}$$

The factor $\|1+g(t)\|_\infty$ can be bounded by 2 independently of $t \in [0,\delta)$. The second factor we rewrite, using (3.13)

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t)-\eta(Y,s),t-s)\, u^0(Y)\,\eta_Y(Y,s)\,dY\,ds \\
&\qquad - \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t)-\xi(Y,s),t-s)\, u^0(Y)\,\xi_Y(Y,s)\,dY\,ds \\
&= \int_0^t \int_{\mathbb{R}} \{\Phi_\alpha(\eta(X,t)-\eta(Y,s),t-s) \\
&\qquad\qquad - \Phi_\alpha(\xi(X,t)-\xi(Y,s),t-s)\}\,(u^0)'(Y)\,dY\,ds \\
&= \int_0^t \int_{\mathbb{R}} \{\Phi_\alpha(\eta(X,t)-\eta(Y,s),t-s) \\
&\qquad\qquad - \Phi_\alpha(\xi(X,t)-\eta(Y,s),t-s)\}\,(u^0)'(Y)\,dY\,ds \\
&\qquad + \int_0^t \int_{\mathbb{R}} \Phi_\alpha(\xi(X,t)-\eta(Y,s),t-s)\,(u^0)'(Y)\,dY\,ds \\
&\qquad - \int_0^t \int_{\mathbb{R}} \Phi_\alpha(\xi(X,t)-\xi(Y,s),t-s)\,(u^0)'(Y)\,dY\,ds
\end{aligned} \tag{3.26}$$

We estimate the first term on the right-hand side of (3.26):

$$\begin{aligned}
&\Big\|\int_0^t \int_{\mathbb{R}} \{\Phi_\alpha(\eta(X,t)-\eta(Y,s),t-s) - \Phi_\alpha(\xi(X,t)-\eta(Y,s),t-s)\}\,(u^0)'(Y)\,dY\,ds\Big\|_\infty \\
&= \Big\|\int_0^t \int_{\mathbb{R}} \{\Phi_\alpha(\eta(X,t)-y,t-s) - \Phi_\alpha(\xi(X,t)-y,t-s)\} \\
&\qquad \cdot (u^0)'(\eta^{-1}(y,s))(\eta_Y(\eta^{-1}(y,s),s))^{-1}\,dy\,ds\Big\|_\infty \\
&= \Big\|\int_0^t \int_{\mathbb{R}} \{\Phi_\alpha(\eta(X,t)-y,t-s) - \Phi_\alpha(\xi(X,t)-y,t-s)\} \\
&\qquad \cdot (u^0)'(\eta^{-1}(y,s))(1+f(\eta^{-1}(y,s),s))^{-1}\,dy\,ds\Big\|_\infty \\
&\leq \frac{\|(u^0)'\|_\infty}{\gamma} \int_0^t \int_{\mathbb{R}} \big|\Phi_\alpha(\eta(X,t)-y,t-s) - \Phi_\alpha(\xi(X,t)-y,t-s)\big|\,dy\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\|(u^0)'\|_\infty}{\gamma} \int_0^t \mathrm{TV}(\Phi_\alpha(\cdot,s))\,ds\,|\eta(X,t)-\xi(X,t)| \\
&= \frac{\|(u^0)'\|_\infty}{\gamma} \int_0^t \int_{\mathbb{R}} |\Phi_{\alpha,x}(y,s)|\,dy\,ds \left| \int_{-\infty}^{X} (f(Z,t)-g(Z,t))\,dZ \right| \\
&\le \frac{\|(u^0)'\|_\infty}{\gamma} \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))} \|f-g\|_{\mathbf{B}}
\end{aligned}
$$

where we have used that $1+f(Z,s) > \gamma$ for $s \in [0,\delta)$; that $\mathrm{TV}(v) = \int |v_x| dx$ for functions $v \in W^{1,1}(\mathbb{R})$, and the definitions of η, ξ in (3.22) and (3.23) respectively. The last expression is bounded since $u^0 \in C_b^1(\mathbb{R})$ and $\Phi_{\alpha,x} \in L^1(\mathbb{R}\times(0,T))$ by (3.6). Furthermore, $\|(u^0)'\|_\infty$ and $\|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))}$ are constants not depending on t.

In order to estimate the second and third term on the right-hand side of (3.26), we integrate by parts in Y again and substitute $y = \eta(Y,s)$ and $z = \xi(Y,s)$ to obtain

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}} \Phi_\alpha(\xi(X,t)-\eta(Y,s),t-s)\,(u^0)'(Y)\,dY\,ds \right. \\
&\qquad \left. - \int_0^t \int_{\mathbb{R}} \Phi_\alpha(\xi(X,t)-\xi(Y,s),t-s)\,(u^0)'(Y)\,dY\,ds \right| \\
&= \left| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t)-\eta(Y,s),t-s)\,u^0(Y)\,\eta_Y(Y,s)\,dY\,ds \right. \\
&\qquad \left. - \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t)-\xi(Y,s),t-s)\,u^0(Y)\,\xi_Y(Y,s)\,dY\,ds \right| \\
&= \left| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t)-y,t-s)\,u^0(\eta^{-1}(y,s),s)\,dy\,ds \right. \\
&\qquad \left. - \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t)-z,t-s)\,u^0(\xi^{-1}(z,s),s)\,dz\,ds \right| \\
&= \left| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t)-y,t-s)\,(u^0(\eta^{-1}(y,s),s)-u^0(\xi^{-1}(y,s),s))\,dy\,ds \right| \\
&\le \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))} \|(u^0)'\|_\infty \sup_{t\in[0,\delta)} \|\eta^{-1}(t)-\xi^{-1}(t)\|_\infty
\end{aligned}
$$

We rewrite $\eta^{-1}(x,t)-\xi^{-1}(x,t)$ as in [2]:

$$
\begin{aligned}
\eta^{-1}(x,t)-\xi^{-1}(x,t) &= -\int_{-\infty}^{\eta^{-1}(x,t)} f(z,t)\,dz + \int_{-\infty}^{\xi^{-1}(x,t)} g(z,t)\,dz \\
&= -\int_{-\infty}^{\eta^{-1}(x,t)} f(z,t)\,dz + \int_{-\infty}^{\eta^{-1}(x,t)} g(z,t)\,dz
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{-\infty}^{\eta^{-1}(x,t)} g(z,t)\,dz + \int_{-\infty}^{\xi^{-1}(x,t)} g(z,t)\,dz \\
&= - \int_{-\infty}^{\eta^{-1}(x,t)} (f(z,t) - g(z,t))\,dz + \int_{\eta^{-1}(x,t)}^{\xi^{-1}(x,t)} g(z,t)\,dz.
\end{aligned}
$$

We take absolute values and obtain

$$
\begin{aligned}
\left|\eta^{-1}(x,t) - \xi^{-1}(x,t)\right| \le & \left| \int_{-\infty}^{\eta^{-1}(x,t)} (f(z,t) - g(z,t))\,dz \right| \\
& + \left|\eta^{-1}(x,t) - \xi^{-1}(x,t)\right| \|g(t)\|_\infty.
\end{aligned}
$$

Now we subtract the second term on the right-hand side of the above equation

$$
\left(1 - \|g(t)\|_\infty\right) \left|\eta^{-1}(x,t) - \xi^{-1}(x,t)\right| \le \left| \int_{-\infty}^{\eta^{-1}(x,t)} (f(z,t) - g(z,t))\,dz \right|.
$$

We take the supremum over all $x \in \mathbb{R}$ and $t \in [0,\delta)$, use that $\|g(t)\|_\infty < 1 - \gamma$ for $t \in [0,\delta)$ and that η is a diffeomorphism, so

$$
\sup_{t\in[0,\delta)} \|\eta^{-1}(t) - \xi^{-1}(t)\|_\infty \le \frac{1}{\gamma}\|f - g\|_{\mathbf{B}}. \tag{3.27}
$$

Thus we can finally estimate (3.26) by

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(X,t) - \eta(Y,s), t-s)\, u^0(Y)\, \eta_Y(Y,s)\, dY\, ds \right. \\
&\left. - \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\xi(X,t) - \xi(Y,s), t-s)\, u^0(Y)\, \xi_Y(Y,s)\, dY\, ds \right| \le C_1 \|f - g\|_{\mathbf{B}}
\end{aligned} \tag{3.28}
$$

with

$$
C_1 = 2\frac{\|(u^0)'\|_\infty}{\gamma} \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))}
$$

Combining (3.25) and (3.28),

$$
\|F(f)(t) - F(g)(t)\|_\infty \le C_2 \|f - g\|_{\mathbf{B}}
$$

with

$$
C_2 = \left(\frac{4}{\gamma} \|(u^0)'\|_\infty + \|u^0\|_\infty \right) \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))}.
$$

C_2 is independent of time and therefore we can take the supremum over all $t \in [0,\delta)$ to obtain

$$
\sup_{t\in[0,\delta)} \|F(f)(t) - F(g)(t)\|_\infty \le C_2 \|f - g\|_{\mathbf{B}}. \tag{3.29}
$$

Next we estimate

$$\sup_{X\in\mathbb{R}}\left|\int_{-\infty}^{X}(F(f)(Z,t)-F(g)(Z,t))\,dZ\right|.$$

Observe that for $h \in U$, where we denote $\beta(X,t) := X + \int_{-\infty}^{X} h(Z,t)\,dZ$, it holds

$$\begin{aligned}
F(h)(X,t) &= \frac{d}{dX}\int_0^t\int_{\mathbb{R}}\Phi_\alpha\big(\beta(X,t)-y,t-s\big)\,u^0(\beta^{-1}(y,s))\,dy\,ds\\
&= \frac{d}{dX}\int_0^t\int_{\mathbb{R}}\Phi_\alpha\big(\beta(X,t)-\beta(Y,t),t-s\big)\,u^0(\beta^{-1}(\beta(Y,t),s))\beta_Y(Y,t)\,dY\,ds\\
&= \frac{d}{dX}\int_0^t\int_{\mathbb{R}}\Phi_\alpha\left(\int_Y^X(1+h(Z,t))\,dZ,t-s\right)u^0(\beta^{-1}(\beta(Y,t),s))(1+h(Y,t))\,dY\,ds.
\end{aligned}$$

Again, $h \in U$ implies $1+h > \gamma > 0$ and therefore $\int_Y^X(1+h(Z,t))\,dZ$ is infinite when $X \to \pm\infty$. Moreover, Φ_α vanishes at $\pm\infty$ and u^0 is bounded in L^∞, thus we can apply the fundamental theorem of calculus to get

$$\begin{aligned}
&\int_{-\infty}^{X} F(h)(Z,t)\,dZ\\
&= \int_0^t\int_{\mathbb{R}}\Phi_\alpha\left(\int_Y^X(1+h(Z,t))\,dZ,t-s\right)u^0(\beta^{-1}(\beta(Y,t),s))(1+h(Y,t))\,dY\,ds\\
&\qquad= \int_0^t\int_{\mathbb{R}}\Phi_\alpha\big(\beta(X,t)-\beta(Y,s),t-s\big)\,u^0(Y)\,\beta_Y(Y,s)\,dY\,ds.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{-\infty}^{X}(F(f)(Z,t)-F(g)(Z,t))\,dZ\\
&\qquad= \int_0^t\int_{\mathbb{R}}\Phi_\alpha\big(\eta(X,t)-\eta(Y,s),t-s\big)\,u^0(Y)\,\eta_Y(Y,s)\,dY\,ds\\
&\qquad\quad- \int_0^t\int_{\mathbb{R}}\Phi_\alpha\big(\xi(X,t)-\xi(Y,s),t-s\big)\,u^0(Y)\,\xi_Y(Y,s)\,dY\,ds. \qquad (3.30)
\end{aligned}$$

Equation (3.30) looks like the left-hand side of (3.26) with Φ_α replaced by $\Phi_{\alpha,x}$. So estimating it follows along similar lines as when estimating (3.26) except that we do not need to integrate by parts. We rewrite (3.30) in the following way

$$\begin{aligned}
&\int_{-\infty}^{X} (F(f)(Z,t) - F(g)(Z,t))\, dZ \\
&= \int_0^t \int_{\mathbb{R}} \{\Phi_\alpha\big(\eta(X,t) - y, t-s\big) - \Phi_\alpha\big(\xi(X,t) - y, t-s\big)\}\, u^0(\eta^{-1}(y,s))\, dy\, ds \\
&\qquad + \int_0^t \int_{\mathbb{R}} \Phi_\alpha\big(\xi(X,t) - y, t-s\big)\, u^0(\eta^{-1}(y,s))\; dy\, ds \\
&\qquad\qquad - \int_0^t \int_{\mathbb{R}} \Phi_\alpha\big(\xi(X,t) - y, t-s\big)\, u^0(\xi^{-1}(y,s))\; dy\, ds
\end{aligned}$$

The first term on the right-hand side can be estimated as follows:

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}} \{\Phi_\alpha\big(\eta(X,t) - y, t-s\big) - \Phi_\alpha\big(\xi(X,t) - y, t-s\big)\}\, u^0(\eta^{-1}(y,s))\, dy\, ds \right| \\
&\quad \le \|u^0\|_\infty \int_0^t \int_{\mathbb{R}} \big|\Phi_\alpha\big(\eta(X,t) - y, t-s\big) - \Phi_\alpha\big(\xi(X,t) - y, t-s\big)\big|\, dy\, ds \\
&\quad \le \|u^0\|_\infty \int_0^t \mathrm{TV}(\Phi_\alpha(\cdot,s))\, ds\, |\eta(X,t) - \xi(X,t)| \\
&\quad \le \|u^0\|_\infty \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))} \left| \int_{-\infty}^{X} (f(Z,t) - g(Z,t)\, dZ \right| \\
&\quad \le \|u^0\|_\infty \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))} \|f - g\|_{\mathbf{B}}.
\end{aligned}$$

For the second and the third term we have:

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}} \Phi_\alpha\big(\xi(X,t) - y, t-s\big)\, (u^0(\eta^{-1}(y,s)) - u^0(\xi^{-1}(y,s)))\; dy\, ds \right| \\
&\qquad \le \|\Phi_\alpha\|_{L^1(\mathbb{R}\times(0,T))}\, \|(u^0)'\|_\infty \|\eta^{-1} - \xi^{-1}\|_\infty.
\end{aligned}$$

Employing (3.27), this translates to

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}} \Phi_\alpha\big(\xi(X,t) - y, t-s\big)\, (u^0(\eta^{-1}(y,s)) - u^0(\xi^{-1}(y,s)))\; dy\, ds \right| \\
&\qquad \le \|\Phi_\alpha\|_{L^1(\mathbb{R}\times(0,T))}\, \frac{\|(u^0)'\|_\infty}{\gamma} \|f - g\|_{\mathbf{B}}.
\end{aligned}$$

Again, $\|\Phi_\alpha\|_{L^1(\mathbb{R}\times(0,T))}$ and $\|(u^0)'\|_\infty$ are independent of time. Hence, we have

$$\sup_{X\in\mathbb{R}, t\in[0,\delta)} \left| \int_{-\infty}^{X} (F(f)(Z,t) - F(g)(Z,t))\, dZ \right| \le C_3 \|f - g\|_{\mathbf{B}}, \tag{3.31}$$

where

$$C_3 = \frac{\|\Phi_\alpha\|_{L^1(\mathbb{R}\times(0,T))}}{\gamma} \|(u^0)'\|_\infty + \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))} \|u^0\|_\infty$$

Now we choose $\varepsilon < \delta$ positive so small that the solution of (3.20) stays in U for $t \in [0, \varepsilon)$. □

3.1.2 Global existence

Having shown local well-posedness of (3.1), we continue to show global existence of the solution. This will imply that we can indeed express the solution at any time t as a reparametrization of the initial data (3.9) and furthermore, that if the initial data is smooth, the same will hold for the solution at any time t.

Proposition 3.3 *The solution to (3.1) exists for arbitrary large times* $0 \leq t < \infty$.

Proof We will basically repeat the proof in [2] with the standard filter replaced by the time dependent filter (3.4).

We show that neither the characteristics (3.7) cross in finite time, nor that there exists (X, t) where $\eta_X(X, t) = \infty$ (which would imply that there is a region without characteristics). In other words, we prove that the map η has a global inverse. This can be done by showing that $\eta_X(X, t)$ is bounded away from zero and infinity. Since $\eta_X(X, 0) = 1$ initially, we have that $\eta_X(X, t) \geq 0$ for some time interval $[0, \tau)$. We assume by contradiction that at time τ the characteristics either cross for the first time, i.e. there is $\overline{X} \in \mathbb{R}$ such that $\eta_X(\overline{X}, \tau) = 0$, or that $\eta_X(\overline{X}, t) = \infty$. We consider once more Equation (3.12) and divide it by $\eta_X(\overline{X}, t)$:

$$\frac{1}{\eta_X(\overline{X}, t)} \frac{d}{dt} \eta_X(\overline{X}, t) = \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(\overline{X}, t) - \eta(Y, s), t - s) u^{0,\alpha}(Y) \eta_Y(Y, s) \, dY \, ds,$$

We integrate from $t = 0$ to $t = T_1 < \tau$ to obtain

$$\log |\eta_X(\overline{X}, T_1)| = \int_0^{T_1} \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(\overline{X}, t) - \eta(Y, s), t - s) u^{0,\alpha}(Y) \eta_Y(Y, s) \, dY \, ds \, dt$$

and hence

$$|\eta_X(\overline{X}, T_1)| = \exp\left(\int_0^{T_1} \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(\overline{X}, t) - \eta(Y, s), t - s) u^{0,\alpha}(Y) \eta_Y(Y, s) \, dY \, ds \, dt \right)$$

We estimate the integral in the exponential on the right-hand side:

$$\begin{aligned} &\left| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(\eta(\overline{X}, t) - \eta(Y, s), t - s) u^{0,\alpha}(Y) \eta_Y(Y, s) \, dY \, ds \right| \\ &\qquad \leq \|u^{0,\alpha}\|_\infty \int_0^t \int_{\mathbb{R}} \left| \Phi_{\alpha,x}(\eta(\overline{X}, t) - \eta(Y, s), t - s) \right| \eta_Y(Y, s) \, dY \, ds \\ &\qquad\qquad \leq \|u^{0,\alpha}\|_\infty \int_0^t \int_{\mathbb{R}} \left| \Phi_{\alpha,x}(\eta(\overline{X}, t) - y), t - s) \right| dy \, ds \\ &\qquad\qquad\qquad \leq \|u^{0,\alpha}\|_\infty \left\| \Phi_{\alpha,x} \right\|_{L^1(\mathbb{R} \times (0,T))}. \end{aligned}$$

The change of variable $y = \eta(Y, s)$ is justified, since $\eta(Y, t)$ is invertible in Y for $t < \tau$ by assumption. $\|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))}$ is finite by (3.6). Since also

$$\|u^{0,\alpha}\|_{L^p(\mathbb{R})} \leq \|u^0\|_{L^p(\mathbb{R})}, \quad \forall 1 \leq p \leq \infty, \tag{3.32}$$

which follows from standard properties of the convolution [4], we finally obtain

$$\exp\left(-\|u^0\|_\infty \frac{C\,T_1}{\sqrt{\alpha}}\right) \leq \left|\eta_X(\overline{X}, T_1)\right| \leq \exp\left(\|u^0\|_\infty \frac{C\,T_1}{\sqrt{\alpha}}\right)$$

for all $T_1 < \tau$ and hence by continuity also for $t = \tau$ which is a contradiction to our assumption. □

Remark 3.4 *We have not made use of the differentiability of $u^{0,\alpha}$ in the proof of Proposition 3.3. The only condition on $u^{0,\alpha}$ which we needed is the boundedness in L^∞. Thus if local existence could be shown for initial data $u^{0,\alpha}$ only satisfying $u^{0,\alpha} \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$, global existence would already follow by Proposition 3.3.*

3.2 Convergence to a weak solution of inviscid Burgers' equation

3.2.1 Preliminary estimates

We have shown in the previous section that the solution of (3.1) can be written as

$$u^\alpha(x, t) = u^{0,\alpha}(\eta^{-1}(x, t))$$

This implies together with (3.32) that the maximum of u^α is bounded by the maximum of u^0 for all times and all α, i.e.

$$\|u^\alpha\|_{L^\infty(\mathbb{R}\times(0,T))} \leq \|u^0\|_{L^\infty(\mathbb{R})}. \tag{3.33}$$

Moreover, the reparametrization of the solution u^α in terms of the initial data implies that the total variation is preserved (the proof is similar to the one done in Section 2.1 for filters not depending on time). So,

$$\mathrm{TV}(u^\alpha(\cdot, t)) = \mathrm{TV}(u^{0,\alpha})$$

The total variation of $u^{0,\alpha}$ is bounded by the total variation of u^0 for any α:

$$\begin{aligned}
\mathrm{TV}(u^{0,\alpha}) &= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\leq 1} \int_{\mathbb{R}} u^{0,\alpha}\,\varphi_x\,dx \\
&= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\leq 1} \int_{\mathbb{R}}\int_{\mathbb{R}} u^0(x-y)\,\omega_\alpha(y)\,dy\,\varphi_x(x)\,dx \\
&= \sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\leq 1} \int_{\mathbb{R}}\int_{\mathbb{R}} u^0(x-y)\,\varphi_x(x)\,dx\,\omega_\alpha(y)\,dy \\
&\leq \int_{\mathbb{R}}\left(\sup_{\varphi\in C_0^1(\mathbb{R}),|\varphi|\leq 1} \int_{\mathbb{R}} u^0(x-y)\,\varphi_x(x)\,dx\right)\omega_\alpha(y)\,dy \\
&\leq \mathrm{TV}(u^0)
\end{aligned} \tag{3.34}$$

where the last inequality follows since $\int \omega_\alpha = 1$. Thus, we have

$$\mathrm{TV}(u^\alpha(\cdot,t)) = \mathrm{TV}(u^{0,\alpha}) \leq \mathrm{TV}(u^0). \tag{3.35}$$

Thanks to the bound on the L^1-norm of Φ_α, (3.5), we can also estimate the L^∞-norm of $\overline{u}^\alpha$ in terms of u^α, using (3.33),

$$\begin{aligned}
\|\overline{u}^\alpha(\cdot,t)\|_{L^\infty(\mathbb{R})} &= \left\|\int_0^t\int_{\mathbb{R}} \Phi_\alpha(y,s)\,u^\alpha(\cdot-y,t-s)\,dy\,ds\right\|_{L^\infty} \\
&\leq \int_0^t\int_{\mathbb{R}} \Phi_\alpha(y,s)\,\|u^\alpha(\cdot,t-s)\|_{L^\infty}\,dy\,ds \\
&\leq \int_0^t\int_{\mathbb{R}} \Phi_\alpha(y,s)\,\|u^{0,\alpha}\|_{L^\infty}\,dy\,ds \\
&= (1-e^{-\frac{t}{\alpha}})\,\|u^{0,\alpha}\|_{L^\infty} \leq \|u^{0,\alpha}\|_{L^\infty} \leq \|u^0\|_{L^\infty}.
\end{aligned} \tag{3.36}$$

Furthermore, we have an L^∞-bound on the derivative of the filtered velocity,

$$\begin{aligned}
\|\overline{u}^\alpha_x(\cdot,t)\|_{L^\infty(\mathbb{R})} &= \left\|\int_0^t\int_{\mathbb{R}} \Phi_{\alpha,x}(x-y,t-s)\,u^\alpha(y,s)\,dy\,ds\right\|_{L^\infty(\mathbb{R})} \\
&\leq \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R}\times(0,T))}\,\|u^{0,\alpha}\|_{L^\infty(\mathbb{R})} \\
&\leq \frac{3}{\sqrt{\pi\alpha}}\|u^0\|_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{3.37}$$

where we have employed (3.6) and (3.33).

The total variation of $\overline{u}^\alpha$ can be bounded by the total variation of u^0:

$$\begin{aligned}
\|\overline{u}_x^\alpha(\cdot,t)\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} \Phi_\alpha(y,t-s)\, u_x^\alpha(x-y,s)\, dy\, ds \right| dx \\
&\le \int_0^t \int_{\mathbb{R}} \Phi_\alpha(y,t-s) \int_{\mathbb{R}} |u_x^\alpha(x-y,s)|\, dx\, dy\, ds \\
&= \int_0^t \int_{\mathbb{R}} \Phi_\alpha(y,t-s) \|u_x^\alpha(\cdot,s)\|_{L^1(\mathbb{R})}\, dy\, ds \\
&\le \mathrm{TV}(u^0),
\end{aligned} \tag{3.38}$$

where we have used (3.5) and (3.35).

We continue to bound the L^1-norm of the second derivative of $\overline{u}^\alpha$. Using substitution, we have

$$\begin{aligned}
\overline{u}_{xx}^\alpha(x,t) &= \frac{d}{dx} \int_0^t \int_{\mathbb{R}} \Phi_\alpha(y,t-s)\, u_x^\alpha(x-y,s)\, dy\, ds \\
&= \frac{d}{dx} \int_0^t \int_{\mathbb{R}} \Phi_\alpha(x-y,t-s)\, u_x^\alpha(y,s)\, dy\, ds \\
&= \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(x-y,t-s)\, u_x^\alpha(y,s)\, dy\, ds
\end{aligned}$$

Thus $\|\overline{u}_{xx}^\alpha(\cdot,t)\|_{L^1(\mathbb{R})}$ can be bounded as follows, using (3.6), (3.35) and Tonelli's Theorem

$$\begin{aligned}
\|\overline{u}_{xx}^\alpha(\cdot,t)\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} \Phi_{\alpha,x}(y,t-s)\, u_x^\alpha(x-y,s)\, dy\, ds \right| dx \\
&\le \int_0^t \int_{\mathbb{R}} |\Phi_{\alpha,x}(y,t-s)| \int_{\mathbb{R}} |u_x^\alpha(x-y,s)|\, dx\, dy\, ds \\
&\le \int_0^t \int_{\mathbb{R}} |\Phi_{\alpha,x}(y,t-s)|\, \|u_x^\alpha(\cdot,s)\|_{L^1(\mathbb{R})}\, dy\, ds \\
&\le \|\Phi_{\alpha,x}\|_{L^1(\mathbb{R})}\, \|u_x^{0,\alpha}\|_{L^1(\mathbb{R})} \\
&\le \frac{3}{\sqrt{\pi\alpha}} \mathrm{TV}(u^0).
\end{aligned} \tag{3.39}$$

In order to estimate the L^1-norm of u_t^α, we consider the first equation in (3.1), subtract the second term on the left-hand side and take the absolute value of both terms,

$$|u_t^\alpha| = |\overline{u}^\alpha|\, |u_x^\alpha|,$$

integrate over the spatial domain and use (3.35) and (3.36)

$$\begin{aligned}
\int_{\mathbb{R}} |u_t^\alpha|\, dx &= \int_{\mathbb{R}} |\overline{u}^\alpha|\, |u_x^\alpha|\, dx \\
&\le \|\overline{u}^\alpha\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |u_x^\alpha|\, dx \\
&\le \|u^0\|_{L^\infty(\mathbb{R})} \mathrm{TV}(u^0).
\end{aligned} \tag{3.40}$$

The L^1-norm of $\overline{u}_t^\alpha$ can be estimated by the L^1-norm of u_t^α

$$\begin{aligned}\int_{\mathbb{R}} |\overline{u}_t^\alpha(x,t)|\,dx &= \int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} \Phi_\alpha(y,s)\, u_t^\alpha(x-y,t-s)\,dy\,ds \right| dx \\ &\le \sup_{t\in(0,T)} \|u_t^\alpha(\cdot,t)\|_{L^1(\mathbb{R})} \|\Phi_\alpha\|_{L^1(\mathbb{R}\times(0,T))} \\ &\le \|u^0\|_{L^\infty(\mathbb{R})} \mathrm{TV}(u^0). \end{aligned} \tag{3.41}$$

In summary, we have the following bounds for u^α and $\overline{u}^\alpha$:

Lemma 3.5 *The solution u^α to Equations (3.1) and the filtered velocity $\overline{u}^\alpha$ satisfy for all $t \in [0,T)$:*

$$\|u^\alpha(\cdot,t)\|_{L^\infty}, \|\overline{u}^\alpha(\cdot,t)\|_{L^\infty} \le \|u^0\|_{L^\infty(\mathbb{R})}, \tag{3.42a}$$

$$TV(u^\alpha(\cdot,t)), TV(\overline{u}^\alpha(\cdot,t)) \le TV(u^0), \tag{3.42b}$$

$$\|\overline{u}_x^\alpha(\cdot,t)\|_{L^\infty} \le \frac{3}{\sqrt{\pi\alpha}} \|u^0\|_{L^\infty(\mathbb{R})}, \tag{3.42c}$$

$$\|\overline{u}_{xx}^\alpha(\cdot,t)\|_{L^1} \le \frac{3}{\sqrt{\pi\alpha}} TV(u^0), \tag{3.42d}$$

$$\|u_t^\alpha(\cdot,t)\|_{L^1(\mathbb{R})}, \|\overline{u}_t^\alpha(\cdot,t)\|_{L^1(\mathbb{R})} \le \|u^0\|_{L^\infty(\mathbb{R})} TV(u^0). \tag{3.42e}$$

3.2.2 Convergence to a weak solution of inviscid Burgers' equation

In order to prove convergence in $C([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}))$ of u^α to a limit function u as $\alpha \to 0$, we need another estimate, namely

Lemma 3.6 *The solution u^α to Equations (3.1) and the filtered velocity $\overline{u}^\alpha$ are uniformly bounded in $C([0,\infty); L^1_{loc}(\mathbb{R}))$:*

$$\|u^\alpha(\cdot,t+k) - u^\alpha(\cdot,t)\|_{L^1(\mathbb{R})} \le \|u^0\|_{L^\infty(\mathbb{R})}\, TV(u^0)\, k, \quad \forall k > 0, \tag{3.43}$$

and

$$\|\overline{u}^\alpha(\cdot,t+k) - \overline{u}^\alpha(\cdot,t)\|_{L^1(\mathbb{R})} \le \|u^0\|_{L^\infty(\mathbb{R})} TV(u^0)\, k, \quad \forall k > 0. \tag{3.44}$$

Proof We use the fundamental theorem of calculus and the bound on the L^1-norm of u_t^α, (3.42e):

$$\begin{aligned}\int_{\mathbb{R}} |u^\alpha(x,t+k) - u^\alpha(x,t)|\,dx &\le \int_{\mathbb{R}} \int_t^{t+k} |u_t^\alpha(x,s)|\,ds\,dx \\ &\le \int_t^{t+k} \|u^0\|_{L^\infty(\mathbb{R})}\, \mathrm{TV}(u^0)\,ds \\ &= \|u^0\|_{L^\infty(\mathbb{R})}\, \mathrm{TV}(u^0)\, k. \end{aligned}$$

In order to estimate $\|\overline{u}^\alpha(\cdot, t+k) - \overline{u}^\alpha(\cdot, t)\|_{L^1(\mathbb{R})}$, we proceed similarly,

$$\begin{aligned}\int_{\mathbb{R}} |\overline{u}^\alpha(x, t+k) - \overline{u}^\alpha(x,t)|\, dx &\le \int_{\mathbb{R}} \int_t^{t+k} |\overline{u}_t^\alpha(x,s)|\, ds\, dx \\ &\le k \sup_{s\in(t,t+k)} \int_{\mathbb{R}} |\overline{u}_t^\alpha(x,s)|\, dx \\ &\le \|u^0\|_{L^\infty(\mathbb{R})} \mathrm{TV}(u^0)\, k. \qquad \square\end{aligned}$$

Using Lemma 3.5 and 3.6, we may prove

Proposition 3.7 *Let u^α, $\overline{u}^\alpha$ solve (3.1). Then, as $\alpha \to 0$, passing to a subsequence if necessary, there exists a function $u(x,t)$ such that*

$$u^\alpha, \overline{u}^\alpha \to u \text{ in } C\big((0,T); L^1_{loc}(\mathbb{R})\big).$$

Moreover, the function u is an element of $BV(\mathbb{R})$.

Proof The fact that $u^\alpha \to u$, $\overline{u}^\alpha \to \widetilde{u}$ in $C((0,T); L^1_{\mathrm{loc}}(\mathbb{R}))$ as $\alpha \to 0$ follows from estimates (3.42a), (3.42b), (3.43) and (3.44) by an application of Kolmogorov's Compactness Theorem (see e.g. Theorem A.8 in [12]). That actually u, $\widetilde{u} \in BV(\mathbb{R})$ follows from Helly's theorem (see Theorem A.7 in [12]). It remains to prove that $u = \widetilde{u}$ almost everywhere. This will be done by showing that for every $t \in (0,T)$, $\Omega \subset \mathbb{R}$ bounded

$$\|\overline{u}^\alpha(\cdot,t) - u(\cdot,t)\|_{L^1(\Omega)} \to 0, \quad \alpha \to 0,$$

which will imply that the limits u and $\widetilde{u}$ agree almost everywhere. (For $t = 0$ this is obvious since $\overline{u}^\alpha(x,0) = u^{\alpha,0}(x)$ We start by rewriting $\overline{u}^\alpha(x,t)$ in a different way (using substitution in the integral)

$$\begin{aligned}\overline{u}^\alpha(x,t) &= \int_0^t \int_{\mathbb{R}} e^{-\frac{t-s}{\alpha}} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{u^\alpha(y,s)}{\alpha}\, dy\, ds \\ &= \int_0^t \int_{\mathbb{R}} e^{-\frac{s}{\alpha}} \frac{1}{\sqrt{4\pi s}} e^{-\frac{y^2}{4s}} \frac{u^\alpha(x-y, t-s)}{\alpha}\, dy\, ds \\ &= \int_0^{t/\alpha} \int_{\mathbb{R}} e^{-r} \frac{1}{\sqrt{4\pi\alpha r}} e^{-\frac{y^2}{4\alpha r}} u^\alpha(x-y, t-\alpha r)\, dy\, dr \\ &= \int_0^{t/\alpha} \int_{\mathbb{R}} e^{-r} \frac{1}{\sqrt{4\pi}} e^{-\frac{z^2}{4}} u^\alpha(x - \sqrt{\alpha r} z, t - \alpha r)\, dz\, dr \\ &= \int_0^{\infty} \int_{\mathbb{R}} e^{-r} \frac{1}{\sqrt{4\pi}} e^{-\frac{z^2}{4}} \chi_{(0,t/\alpha)}(r) u^\alpha(x - \sqrt{\alpha r} z, t - \alpha r)\, dz\, dr. \qquad (3.45)\end{aligned}$$

Note that the integrand is uniformly bounded for all α by

$$e^{-r} \frac{1}{\sqrt{4\pi}} e^{-\frac{z^2}{4}} \|u^0\|_{L^\infty(\mathbb{R})},$$

which is integrable in $L^1(\mathbb{R} \times (0, \infty) \times \Omega)$ for all $\Omega \subset \mathbb{R}$ bounded. We denote

$$\psi(r,z) := e^{-r} \frac{1}{\sqrt{4\pi}} e^{-\frac{z^2}{4}}.$$

Adding and subtracting terms, we can write the difference $\overline{u}^\alpha(x,t) - u(x,t)$ as

$$\begin{aligned}
\overline{u}^\alpha(x,t) - u(x,t) &= \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \psi(r,z) u^\alpha(x - \sqrt{\alpha r} z, t - \alpha r)\, dz\, dr - u(x,t) \\
&= \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \psi(r,z)(u^\alpha(x - \sqrt{\alpha r} z, t - \alpha r) - u^\alpha(x - \sqrt{\alpha r} z, t))\, dz\, dr \\
&+ \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \psi(r,z)(u^\alpha(x - \sqrt{\alpha r} z, t) - u(x - \sqrt{\alpha r} z, t))\, dz\, dr \\
&+ \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \psi(r,z)(u(x - \sqrt{\alpha r} z, t) - u(x,t))\, dz\, dr \\
&- e^{-\frac{t}{\alpha}} u(x,t)
\end{aligned}$$

Hence for $\Omega \subset \mathbb{R}$ bounded, we have

$$\begin{aligned}
&\int_\Omega |\overline{u}^\alpha(x,t) - u(x,t)|\, dx \\
&\quad\le \int_\Omega \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \psi(r,z) |u^\alpha(x - \sqrt{\alpha r} z, t - \alpha r) - u^\alpha(x - \sqrt{\alpha r} z, t)|\, dz\, dr\, dx \\
&\quad\quad + \int_\Omega \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \psi(r,z) |u^\alpha(x - \sqrt{\alpha r} z, t) - u(x - \sqrt{\alpha r} z, t)|\, dz\, dr\, dx \\
&\quad\quad\quad + \int_\Omega \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \psi(r,z) |u(x - \sqrt{\alpha r} z, t) - u(x,t)|\, dz\, dr\, dx \\
&\quad\quad\quad\quad + e^{-\frac{t}{\alpha}} \int_\Omega |u(x,t)|\, dx \\
&\quad\quad\quad\quad\quad := A + B + C + D.
\end{aligned}$$

We estimate the terms A, B, C, D separately. For A we have, by (3.43) and Fubini's Theorem,

$$\begin{aligned}
A &= \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \psi(r,z) \int_\Omega |u^\alpha(x - \sqrt{\alpha r} z, t - \alpha r) - u^\alpha(x - \sqrt{\alpha r} z, t)|\, dx\, dz\, dr \\
&\le \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \psi(r,z) M \alpha r\, dz\, dr \\
&= M \int_0^{\frac{t}{\alpha}} \alpha r e^{-r}\, dr \\
&= M\alpha - M(\alpha + t)\, e^{-\frac{t}{\alpha}} \xrightarrow{\alpha \to 0} 0.
\end{aligned}$$

For B we have, again using Fubini's Theorem

$$\begin{aligned} B &= \int_\Omega \int_0^\infty \int_{\mathbb{R}} \psi(r,z)\chi_{(0,t/\alpha)}(r)|u^\alpha(x-\sqrt{\alpha r}z,t) - u(x-\sqrt{\alpha r}z,t)|\,dz\,dr\,dx \\ &= \int_0^\infty \int_{\mathbb{R}} \psi(r,z)\chi_{(0,t/\alpha)}(r) \int_\Omega |u^\alpha(x-\sqrt{\alpha r}z,t) - u(x-\sqrt{\alpha r}z,t)|\,dx\,dz\,dr. \end{aligned}$$

The integrand is uniformly bounded by an integrable function for all α, therefore we can apply Lebesgue's Dominated Convergence Theorem and pass the limit under the integral signs. Since $u^\alpha \to u$ in $C((0,T);L^1_{\text{loc}}(\mathbb{R}))$ for $\alpha \to 0$, we have

$$\begin{aligned} &\chi_{(0,t/\alpha)}(r) \int_\Omega |u^\alpha(x-\sqrt{\alpha r}z,t) - u(x-\sqrt{\alpha r}z,t)|\,dx \\ &\qquad\qquad \le \int_\Omega |u^\alpha(x-\sqrt{\alpha r}z,t) - u(x-\sqrt{\alpha r}z,t)|\,dx \xrightarrow{\alpha\to 0} 0 \end{aligned}$$

which implies $B \to 0$ as $\alpha \to 0$. We write C as

$$\begin{aligned} C &= \int_\Omega \int_0^\infty \int_{\mathbb{R}} \psi(r,z)\chi_{(0,t/\alpha)}(r)|u(x-\sqrt{\alpha r}z,t) - u(x,t)|\,dz\,dr\,dx \\ &= \int_0^\infty \int_{\mathbb{R}} \psi(r,z)\chi_{(0,t/\alpha)}(r) \int_\Omega |u(x-\sqrt{\alpha r}z,t) - u(x,t)|\,dx\,dz\,dr. \end{aligned}$$

Again, since the integrand is uniformly bounded by an integrable function, we can apply Lebesgue's Theorem and pass to the limit inside the integrals. Because $u \in \mathrm{BV}(\mathbb{R})$, it is continuous everywhere except for a set of measure zero and we have for every fixed r and $z \in \mathbb{R}$

$$\lim_{\alpha\to 0} |u(x-\sqrt{\alpha r}z,t) - u(x,t)| = 0, \qquad \text{a.e. } x \in \mathbb{R}.$$

Therefore

$$\begin{aligned} &\lim\nolimits_{\alpha\to 0} \int_0^\infty \int_{\mathbb{R}} \psi(r,z)\chi_{(0,t/\alpha)}(r) \int_\Omega |u(x-\sqrt{\alpha r}z,t) - u(x,t)|\,dx\,dz\,dr \\ &= \int_0^\infty \int_{\mathbb{R}} \psi(r,z) \lim_{\alpha\to 0}\{\chi_{(0,t/\alpha)}(r) \int_\Omega |u(x-\sqrt{\alpha r}z,t) - u(x,t)|\,dx\}\,dz\,dr \\ &\le \int_0^\infty \int_{\mathbb{R}} \psi(r,z) \int_\Omega \lim_{\alpha\to 0} |u(x-\sqrt{\alpha r}z,t) - u(x,t)|\,dx\,dz\,dr \\ &= 0. \end{aligned}$$

That D converges to zero as $\alpha \to 0$ is obvious, since $u \in L^1_{\text{loc}}(\mathbb{R})$ and $e^{-t/\alpha} \to 0$ as $\alpha \to 0$ for every $t > 0$. Hence we have shown that the limits u and $\widetilde{u}$ agree. □

Now we will show that the limit function u is actually a weak solution of inviscid Burgers' equation.

Proposition 3.8 *The function u from Proposition 3.7 obtained as a limit of u^α in $C((0,T);L^1_{loc}(\mathbb{R}))$, as $\alpha \to 0$, is actually a weak solution of the inviscid Burgers' equation with initial data u^0, i.e. it satisfies for all $\varphi \in C_c^{2,2}(\mathbb{R} \times [0,T))$*

$$\int_0^T \int_{\mathbb{R}} u\,\varphi_t + \frac{u^2}{2}\varphi_x\,dx\,dt + \int_{\mathbb{R}} u^0(x)\,\varphi(x,0)\,dx = 0 \tag{3.46}$$

Proof We know by Proposition 3.7 that the limits of $\overline{u}^\alpha$ and u^α agree, so it is enough to show that $\overline{u}^\alpha$ converges to a weak solution of inviscid Burgers' equation as $\alpha \to 0$. We insert the second equation (3.1b) of (3.1) into the first equation (3.1a) to obtain

$$\overline{u}^\alpha_t + \alpha(\overline{u}^\alpha_{tt} - \overline{u}^\alpha_{xxt}) + \overline{u}^\alpha(\overline{u}^\alpha_x + \alpha(\overline{u}^\alpha_{xt} - \overline{u}^\alpha_{xxx})) = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{3.47a}$$

$$\overline{u}^\alpha(x,0) = u^{0,\alpha}(x), \quad x \in \mathbb{R}, \tag{3.47b}$$

$$u^{0,\alpha}(x) = (\omega_\alpha * u^0)(x), \quad x \in \mathbb{R}, \tag{3.47c}$$

We rewrite the first equation, bringing all the terms which are multiplied by α to the right-hand side:

$$\overline{u}^\alpha_t + \overline{u}^\alpha \overline{u}^\alpha_x = \alpha(-\overline{u}^\alpha_{tt} + \overline{u}^\alpha_{xxt} - \overline{u}^\alpha \overline{u}^\alpha_{xt} + \overline{u}^\alpha \overline{u}^\alpha_{xxx}).$$

Now we multiply by a test function $\varphi \in C_c^{2,2}(\mathbb{R} \times [0,T))$ and integrate over $\mathbb{R} \times (0,T)$

$$\int_0^T \int_{\mathbb{R}} \overline{u}^\alpha_t \varphi + \left(\frac{\overline{u}^{\alpha 2}}{2}\right)_x \varphi\,dx\,dt = \alpha \int_0^T \int_{\mathbb{R}} (-\overline{u}^\alpha_{tt} + \overline{u}^\alpha_{xxt} - \overline{u}^\alpha \overline{u}^\alpha_{xt} + \overline{u}^\alpha \overline{u}^\alpha_{xxx})\varphi\,dx\,dt \tag{3.48}$$

Integrating the left-hand side of this equation by parts, it becomes

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \overline{u}^\alpha_t \varphi + \left(\frac{\overline{u}^{\alpha 2}}{2}\right)_x \varphi\,dx\,dt \\
&\qquad = -\int_0^T \int_{\mathbb{R}} u^\alpha\,\varphi_t + \frac{u^{\alpha 2}}{2}\varphi_x\,dx\,dt - \int_{\mathbb{R}} u^{\alpha,0}(x)\,\varphi(x,0)\,dx \\
&\qquad\qquad \xrightarrow{\alpha \to 0} -\int_0^T \int_{\mathbb{R}} u\,\varphi_t + \frac{u^2}{2}\varphi_x\,dx\,dt - \int_{\mathbb{R}} u^0(x)\,\varphi(x,0)\,dx
\end{aligned}$$

by the convergence of $\overline{u}^\alpha$ to u in $C((0,T);L^1_{\text{loc}}(\mathbb{R}))$ and $u^{\alpha,0}$ to u^0 in $L^1(\mathbb{R})$. This is the left-hand side of (3.46). Hence we have to show that all the terms on the right-hand side of (3.48) converge to zero as α goes to zero. We integrate the first two terms on the right-hand side twice, three times respectively, by parts to obtain

$$\begin{aligned}
\alpha \int_0^T \int_{\mathbb{R}} (-\overline{u}^\alpha_{tt} + \overline{u}^\alpha_{xxt})\varphi\,dx\,dt &= \alpha \int_0^T \int_{\mathbb{R}} -\overline{u}^\alpha \varphi_{tt} - \overline{u}^\alpha \varphi_{xxt}\,dx\,dt \\
&\quad + \alpha \int_{\mathbb{R}} u^{\alpha,0}(x)(\varphi_t(x,0) + \varphi_{xx}(x,0)) - \overline{u}^\alpha_t(x,0)\varphi(x,0)\,dx
\end{aligned}$$

By (3.42a) $\overline{u}^\alpha$ is uniformly bounded, which gives

$$\alpha \int_0^T \int_{\mathbb{R}} -\overline{u}^\alpha \varphi_{tt} - \overline{u}^\alpha \varphi_{xxt}\, dx\, dt \\ \leq \alpha \|u^0\|_{L^\infty} \big(\|\varphi_{tt}\|_{L^1(\mathrm{supp}(\varphi))} + \|\varphi_{xxt}\|_{L^1(\mathrm{supp}(\varphi))} \big) \xrightarrow{\alpha\to 0} 0.$$

For the boundary terms, we have, by (3.32) and (3.42e)

$$\alpha \int_{\mathbb{R}} u^{\alpha,0}(x)(\varphi_t(x,0) + \varphi_{xx}(x,0)) - \overline{u}^\alpha_t(x,0)\varphi(x,0)\, dx \\ \leq \|u^0\|_{L^\infty} \big(\|\varphi_t(\cdot,0)\|_{L^1(\mathbb{R})} + \|\varphi_{xx}(\cdot,0)\|_{L^1(\mathbb{R})} + \mathrm{TV}(u^0)\|\varphi\|_{L^\infty} \big) \xrightarrow{\alpha\to 0} 0.$$

Hence

$$\alpha \int_0^T \int_{\mathbb{R}} (-\overline{u}^\alpha_{tt} + \overline{u}^\alpha_{xxt})\varphi\, dx\, dt \xrightarrow{\alpha\to 0} 0.$$

We continue to show that the third term on the right-hand side of (3.48) converges to zero. We integrate by parts in the variable x

$$\begin{aligned} -\alpha \int_0^T \int_{\mathbb{R}} \overline{u}^\alpha \overline{u}^\alpha_{xt} \varphi\, dx\, dt &= \alpha \int_0^T \int_{\mathbb{R}} (\overline{u}^\alpha_x \overline{u}^\alpha_t \varphi + \overline{u}^\alpha \overline{u}^\alpha_t \varphi_x)\, dx\, dt \\ &\leq \alpha \|\overline{u}^\alpha_x\|_{L^\infty} \|\overline{u}^\alpha_t\|_{L^1(\mathbb{R})} \|\varphi\|_{L^\infty} T + \alpha \|\overline{u}^\alpha\|_{L^\infty} \|\overline{u}^\alpha_t\|_{L^1(\mathbb{R})} \|\varphi_x\|_{L^\infty} T \\ &\leq \sqrt{\alpha} C \|u^0\|^2_{L^\infty} \mathrm{TV}(u^0) \|\varphi\|_{L^\infty} T + \alpha \|u^0\|^2_{L^\infty} \mathrm{TV}(u^0) \|\varphi_x\|_{L^\infty} T \xrightarrow{\alpha\to 0} 0, \end{aligned}$$

where we used (3.42a), (3.42c) and (3.42e). Finally, we show that the forth term on the right-hand side of (3.48) converges to zero. Again, we integrate by parts,

$$\begin{aligned} \alpha \int_0^T \int_{\mathbb{R}} \overline{u}^\alpha \overline{u}^\alpha_{xxx} \varphi\, dx\, dt &= -\alpha \int_0^T \int_{\mathbb{R}} (\overline{u}^\alpha_x \overline{u}^\alpha_{xx} \varphi + \overline{u}^\alpha \overline{u}^\alpha_{xx} \varphi_x)\, dx\, dt \\ &= -\alpha \int_0^T \int_{\mathbb{R}} \left\{ \frac{1}{2}((\overline{u}^\alpha_x)^2)_x \varphi + \overline{u}^\alpha \overline{u}^\alpha_{xx} \varphi_x \right\} dx\, dt \\ &= \alpha \int_0^T \int_{\mathbb{R}} \left\{ \frac{1}{2}(\overline{u}^\alpha_x)^2 \varphi_x - \overline{u}^\alpha \overline{u}^\alpha_{xx} \varphi_x \right\} dx\, dt \\ &\leq \alpha \left\{ \frac{1}{2} \|\overline{u}^\alpha_x\|_{L^\infty} \|\overline{u}^\alpha_x\|_{L^1(\mathbb{R})} + \|\overline{u}^\alpha\|_{L^\infty} \|\overline{u}^\alpha_{xx}\|_{L^1(\mathbb{R})} \right\} \|\varphi_x\|_{L^\infty} T \\ &\leq \sqrt{\alpha}\, C\, \|u^0\|_{L^\infty} \mathrm{TV}(u^0) \|\varphi_x\|_{L^\infty} T \xrightarrow{\alpha\to 0} 0, \end{aligned}$$

where we have employed (3.42a), (3.42b), (3.42c) and (3.42d). Thus, we have shown that the right-hand side of (3.48) converges to zero which implies that the limit function u is a weak solution of inviscid Burgers' equation. □

3.2.3 Mass conservation

In contrast to inviscid Burgers' equation and the CFB equation with a time independent, symmetric filter, the solution u^α of (3.1) does in general not satisfy mass conservation. Nevertheless, under the assumption that $\overline{u}^\alpha$, $\overline{u}^\alpha_{xt}$, $(\overline{u}^{\alpha 2})_{xx}$, $\overline{u}^{\alpha 2}_x$ vanish at infinity the difference in mass can be estimated with respect to α to gain mass conservation in the limit $\alpha \to 0$. We therefore consider again the equation for $\overline{u}^\alpha$, (3.47),

$$\overline{u}^\alpha_t + \alpha(\overline{u}^\alpha_{tt} - \overline{u}^\alpha_{xxt}) + \overline{u}^\alpha(\overline{u}^\alpha_x + \alpha(\overline{u}^\alpha_{xt} - \overline{u}^\alpha_{xxx})) = 0, \quad (x,t) \in \mathbb{R} \times (0,T).$$

We integrate the equation over the spatial domain $\mathbb{R}$ and use that $\overline{u}^\alpha$ and its derivatives vanish at infinity:

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \overline{u}^\alpha_t + \alpha(\overline{u}^\alpha_{tt} - \overline{u}^\alpha_{xxt}) + \overline{u}^\alpha(\overline{u}^\alpha_x + \alpha(\overline{u}^\alpha_{xt} - \overline{u}^\alpha_{xxx}))\,dx \\ &= \frac{d}{dt}\int_{\mathbb{R}} \overline{u}^\alpha + \alpha\overline{u}^\alpha_t\,dx - \alpha\int_{\mathbb{R}} \overline{u}^\alpha_x\overline{u}^\alpha_t + \overline{u}^\alpha\overline{u}^\alpha_{xxx}\,dx \end{aligned}$$

where we have integrated by parts in the second equation. The term $\overline{u}^\alpha\overline{u}^\alpha_{xxx}$ can be written as

$$\overline{u}^\alpha\overline{u}^\alpha_{xxx} = \frac{1}{2}\left(\overline{u}^{\alpha 2}\right)_{xxx} - \frac{3}{2}\left(\overline{u}^{\alpha 2}_x\right)_x$$

which is zero integrated over $\mathbb{R}$ by our assumptions. We integrate the remaining terms over the time domain $(0,T)$

$$\begin{aligned} 0 = \int_{\mathbb{R}} \overline{u}^\alpha(x,T)\,dx - \int_{\mathbb{R}} \overline{u}^{0,\alpha}(x)\,dx \\ + \alpha\left(\int_{\mathbb{R}} \overline{u}^\alpha_t(x,T)\,dx - \int_{\mathbb{R}} \overline{u}^{0,\alpha}_t(x)\,dx - \int_0^T\int_{\mathbb{R}} \overline{u}^\alpha_x\overline{u}^\alpha_t\,dx\,dt\right) \end{aligned}$$

By (3.42e), we have

$$\begin{aligned} \alpha\left(\int_{\mathbb{R}} \overline{u}^\alpha_t(x,T)\,dx - \int_{\mathbb{R}} \overline{u}^{0,\alpha}_t(x)\,dx\right) &\le \alpha\left(\int_{\mathbb{R}} |\overline{u}^\alpha_t(x,T)|\,dx + \int_{\mathbb{R}} |\overline{u}^{0,\alpha}_t(x)|\,dx\right) \\ &\le 2\alpha\|u^0\|_{L^\infty(\mathbb{R})}\,\mathrm{TV}(u^0) \xrightarrow{\alpha\to 0} 0. \end{aligned}$$

Using (3.42e) once more and in addition (3.42b), we obtain

$$\begin{aligned} \alpha\int_0^T\int_{\mathbb{R}} \overline{u}^\alpha_x\,\overline{u}^\alpha_t\,dx\,dt &\le \alpha\int_0^T\int_{\mathbb{R}} |\overline{u}^\alpha_x|\,|\overline{u}^\alpha_t|\,dx\,dt \\ &\le \sqrt{\alpha}\,C\,T\|u^0\|^2_{L^\infty(\mathbb{R})}\,\mathrm{TV}(u^0) \xrightarrow{\alpha\to 0} 0. \end{aligned}$$

Therefore,

$$0 = \lim_{\alpha\to 0}\left(\int_{\mathbb{R}} \overline{u}^\alpha(x,T)\,dx - \int_{\mathbb{R}} \overline{u}^{0,\alpha}(x)\,dx\right) = \int_{\mathbb{R}} u(x,T)\,dx - \int_{\mathbb{R}} u^0(x)\,dx. \tag{3.49}$$

3.2.4 Entropy inequality

Even though (3.1) is not reversible in time, which we achieved by using a filter depending on time, it seems impossible to show that the limit u of the sequence u^α satisfies the entropy inequality. This would be needed to show it is an entropy solution. We will demonstrate that it is not possible to show an entropy inequality for entropies of the form

$$\eta(u) = \frac{u^p}{p}, \quad p \text{ even.} \tag{3.50}$$

As before, we work with Equation (3.47) which we multiply by $\eta'(\overline{u}^\alpha) = \overline{u}^{\alpha(p-1)}$

$$\begin{aligned} 0 &= \overline{u}^{\alpha(p-1)}\overline{u}^\alpha_t + \alpha(\overline{u}^{\alpha(p-1)}\overline{u}^\alpha_{tt} - \overline{u}^{\alpha(p-1)}\overline{u}^\alpha_{xxt}) + \overline{u}^{\alpha p}(\overline{u}^\alpha_x + \alpha(\overline{u}^\alpha_{xt} - \overline{u}^\alpha_{xxx})) \\ &= \eta(\overline{u}^\alpha)_t + q(\overline{u}^\alpha)_x + \alpha(\overline{u}^{\alpha(p-1)}\overline{u}^\alpha_{tt} - \overline{u}^{\alpha(p-1)}\overline{u}^\alpha_{xxt}) + \alpha\overline{u}^{\alpha p}(\overline{u}^\alpha_{xt} - \overline{u}^\alpha_{xxx}) \end{aligned} \tag{3.51}$$

where we have denoted by q the entropy flux satisfying $q'(u) = u\,\eta'(u)$. We bring the terms without factor α to the left-hand side of the equation, multiply by a nonnegative test function $\varphi \in C_c^{3,2}(\mathbb{R} \times (0,T))$ and integrate over the domain $\mathbb{R} \times (0,T)$

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}} (\eta(\overline{u}^\alpha)_t + q(\overline{u}^\alpha)_x)\varphi \, dx\, dt & \\ = \alpha\Bigg(\int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha(p-1)}\overline{u}^\alpha_{tt}\varphi\, dx\, dt - \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha(p-1)}\overline{u}^\alpha_{xxt}\varphi\, dx\, dt & \\ + \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha p}\overline{u}^\alpha_{xt}\varphi\, dx\, dt - \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha p}\overline{u}^\alpha_{xxx}\varphi\, dx\, dt\Bigg) & \\ := A + B + C + D. & \end{aligned} \tag{3.52}$$

We integrate the left-hand side by parts and take the limit $\alpha \to 0$

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}} (\eta(\overline{u}^\alpha)_t + q(\overline{u}^\alpha)_x)\varphi \, dx\, dt &= \int_0^T \int_{\mathbb{R}} \eta(\overline{u}^\alpha)\varphi_t + q(\overline{u}^\alpha)\varphi_x \, dx\, dt \\ &\xrightarrow{\alpha\to 0} \int_0^T \int_{\mathbb{R}} \eta(u)\varphi_t + q(u)\varphi_x \, dx\, dt \end{aligned}$$

since $\overline{u}^\alpha$ is uniformly bounded. If u were an entropy solution, it should be possible to show that the right-hand side of the above equation is greater or equal to zero. We estimate the integrals A, B, C and D. We start with A.

Integrating by parts in the variable t, we get

$$\begin{aligned}\alpha \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha(p-1)} \overline{u}^{\alpha}_{tt} \varphi \, dx \, dt &= -\alpha \int_0^T \int_{\mathbb{R}} (p-1) \overline{u}^{\alpha(p-2)} \overline{u}^{\alpha 2}_t \varphi + \overline{u}^{\alpha(p-1)} \overline{u}^{\alpha}_t \varphi_t \, dx \, dt \\ &= -\alpha \int_0^T \int_{\mathbb{R}} (p-1) \overline{u}^{\alpha(p-2)} \overline{u}^{\alpha 2}_t \varphi + \left(\frac{\overline{u}^{\alpha p}}{p} \right)_t \varphi_t \, dx \, dt \\ &= -\alpha \int_0^T \int_{\mathbb{R}} (p-1) \overline{u}^{\alpha(p-2)} \overline{u}^{\alpha 2}_t \varphi - \frac{\overline{u}^{\alpha p}}{p} \varphi_{tt} \, dx \, dt.\end{aligned}$$

The second term on the right-hand side converges to zero as $\alpha \to 0$ since $\overline{u}^{\alpha}$ is uniformly bounded according to (3.42a) and since $\varphi \in C_c^{3,2}(\mathbb{R} \times (0, T))$. The first term is negative, since $\varphi \geq 0$ and 2 and $p-2$ are even numbers. We consider the integral B

$$\begin{aligned}& - \alpha \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha(p-1)} \overline{u}^{\alpha}_{xxt} \varphi \, dx \, dt \\ &= \alpha \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha(p-1)} \overline{u}^{\alpha}_{xt} \varphi_x + (p-1) \overline{u}^{\alpha(p-2)} \overline{u}^{\alpha}_x \overline{u}^{\alpha}_{xt} \varphi \, dx \, dt \\ &= \alpha \int_0^T \int_{\mathbb{R}} -\overline{u}^{\alpha(p-1)} \overline{u}^{\alpha}_x \varphi_{xt} - (p-1) \overline{u}^{\alpha(p-2)} \overline{u}^{\alpha}_t \overline{u}^{\alpha}_x \varphi_x + \frac{p-1}{2} \overline{u}^{\alpha(p-2)} \left(\overline{u}^{\alpha 2}_x \right)_t \varphi \, dx \, dt \\ &= -\alpha \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha(p-1)} \overline{u}^{\alpha}_x \varphi_{xt} + (p-1) \overline{u}^{\alpha(p-2)} \overline{u}^{\alpha}_t \overline{u}^{\alpha}_x \varphi_x + \frac{p-1}{2} \overline{u}^{\alpha(p-2)} \overline{u}^{\alpha 2}_x \varphi_t \\ &\quad + \frac{(p-1)(p-2)}{2} \overline{u}^{\alpha(p-3)} \overline{u}^{\alpha 2}_x \overline{u}^{\alpha}_t \varphi \, dx \, dt.\end{aligned}$$

Note that the last term disappears if $p = 2$. We bound the terms on the right-hand side: Using (3.42a) and (3.42b), we have

$$\begin{aligned}-\alpha \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha(p-1)} \overline{u}^{\alpha}_x \varphi_{xt} \, dx \, dt &\leq \alpha \, T \, \|\overline{u}^{\alpha}\|^{p-1}_{L^\infty} \|\overline{u}^{\alpha}_x\|_{L^1(\mathbb{R})} \|\varphi_{xt}\|_{L^\infty} \\ &\leq \alpha \, T \, \|u^0\|^{p-1}_{L^\infty} \mathrm{TV}(u^0) \, \|\varphi_{xt}\|_{L^\infty} \xrightarrow{\alpha \to 0} 0. \quad (3.53)\end{aligned}$$

For the second term on the right-hand side, we estimate with (3.42a), (3.42c) and (3.42e)

$$\begin{aligned}-\alpha \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha(p-2)} \overline{u}^{\alpha}_t \overline{u}^{\alpha}_x \varphi_x \, dx \, dt &\leq \alpha \, T \, \|\overline{u}^{\alpha}\|^{p-2}_{L^\infty} \|\overline{u}^{\alpha}_t\|_{L^1(\mathbb{R})} \|\overline{u}^{\alpha}_x\|_{L^\infty} \|\varphi_x\|_{L^\infty} \\ &\leq \sqrt{\alpha} \, C \, T \|u^0\|^{p}_{L^\infty} \mathrm{TV}(u^0) \|\varphi_x\|_{L^\infty} \xrightarrow{\alpha \to 0} 0.\end{aligned}$$

The third term can be bounded using (3.42a), (3.42b) and (3.42c)

$$\begin{aligned}-\alpha \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha(p-2)} \overline{u}^{\alpha 2}_x \varphi_t \, dx \, dt &\leq \alpha \, T \, \|\overline{u}^{\alpha}\|^{p-2}_{L^\infty} \|\overline{u}^{\alpha}_x\|_{L^\infty} \|\overline{u}^{\alpha}_x\|_{L^1(\mathbb{R})} \|\varphi_t\|_{L^\infty} \\ &\leq \sqrt{\alpha} \, C \, T \|u^0\|^{p-1}_{L^\infty} \mathrm{TV}(u^0) \|\varphi_t\|_{L^\infty} \xrightarrow{\alpha \to 0} 0.\end{aligned}$$

Hence we have,

$$B = -\alpha \int_0^T \int_{\mathbb{R}} \frac{(p-1)(p-2)}{2} \overline{u}^{\alpha(p-3)} \overline{u}_x^{\alpha 2} \overline{u}_t^{\alpha} \varphi \, dx \, dt + \mathcal{O}(\sqrt{\alpha})$$

if $p > 2$, and

$$B \xrightarrow{\alpha \to 0} 0$$

if $p = 2$. We continue to estimate C. Integrating by parts, we get

$$\alpha \int_0^T \int_{\mathbb{R}} \overline{u}^{\alpha p} \overline{u}_{xt}^{\alpha} \varphi \, dx \, dt = -\alpha \int_0^T \int_{\mathbb{R}} p \, \overline{u}^{\alpha(p-1)} \overline{u}_t^{\alpha} \overline{u}_x^{\alpha} \varphi + \overline{u}^{\alpha p} \overline{u}_x^{\alpha} \varphi_t \, dx \, dt.$$

We use (3.42a), (3.42c) and (3.42e) to obtain

$$\begin{aligned} -\alpha \int_0^T \int_{\mathbb{R}} p \, \overline{u}^{\alpha(p-1)} \overline{u}_t^{\alpha} \overline{u}_x^{\alpha} \varphi \, dx \, dt &\leq \alpha \, T \|\overline{u}^{\alpha}\|_{L^\infty}^{p-1} \|\overline{u}_x^{\alpha}\|_{L^\infty} \|\overline{u}_t^{\alpha}\|_{L^1(\mathbb{R})} \|\varphi\|_{L^\infty} \\ &\leq \sqrt{\alpha} \, C \, T \|u^0\|_{L^\infty}^{p+1} \mathrm{TV}(u^0) \|\varphi\|_{L^\infty} \xrightarrow{\alpha \to 0} 0. \end{aligned}$$

The other term can be bounded in the same way as (3.53). Thus,

$$C \xrightarrow{\alpha \to 0} 0.$$

For D, we write

$$\overline{u}^{\alpha p} \overline{u}_{xxx}^{\alpha} = \left(\frac{\overline{u}^{\alpha(p+1)}}{p+1} \right)_{xxx} - \frac{3\,p}{2} \overline{u}^{\alpha(p-1)} \left(\overline{u}_x^{\alpha 2} \right)_x - p(p-1) \overline{u}^{\alpha(p-2)} \overline{u}_x^{\alpha 3}$$

Hence

$$\begin{aligned} D &= -\alpha \int_0^T \int_{\mathbb{R}} \left(\frac{\overline{u}^{\alpha(p+1)}}{p+1} \right)_{xxx} - \frac{3\,p}{2} \overline{u}^{\alpha(p-1)} \left(\overline{u}_x^{\alpha 2} \right)_x - p(p-1) \overline{u}^{\alpha(p-2)} \overline{u}_x^{\alpha 3} \varphi \, dx \, dt \\ &= \alpha \int_0^T \int_{\mathbb{R}} \frac{\overline{u}^{\alpha(p+1)}}{p+1} \varphi_{xxx} - \frac{3\,p}{2} \overline{u}^{\alpha(p-1)} \overline{u}_x^{\alpha 2} \varphi_x + p(p-1) \overline{u}^{\alpha(p-2)} \overline{u}_x^{\alpha 3} \varphi \, dx \, dt \end{aligned}$$

The first and the second term on the right-hand side converge to zero as $\alpha \to 0$ by (3.42a), (3.42b) and (3.42c). For the third term we can only show that it is bounded using the estimates derived beforehand. Therefore,

$$C = \alpha \int_0^T \int_{\mathbb{R}} p(p-1) \overline{u}^{\alpha(p-2)} \overline{u}_x^{\alpha 3} \varphi \, dx \, dt + \mathcal{O}(\sqrt{\alpha}).$$

In summary, we have shown,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \eta(\overline{u}^{\alpha}) \varphi_t + q(\overline{u}^{\alpha}) \varphi_x \, dx \, dt \\ &\qquad = \alpha \int_0^T \int_{\mathbb{R}} (p-1)(-\overline{u}^{\alpha(p-2)} \overline{u}_t^{\alpha 2} + p \, \overline{u}^{\alpha(p-2)} \overline{u}_x^{\alpha 3}) \varphi \\ &\qquad\qquad - \frac{(p-1)(p-2)}{2} \overline{u}^{\alpha(p-3)} \overline{u}_x^{\alpha 2} \overline{u}_t^{\alpha} \varphi \, dx \, dt + \mathcal{O}(\sqrt{\alpha}). \end{aligned} \tag{3.54}$$

(The third term in the integral disappears if $p = 2$.) Unfortunately, the first term on the right-hand side has the wrong sign and the second and the third term do not have a sign. Hence, still if we could show that either of them converged to zero as α converges to zero, we wouldn't get the desired entropy inequality.

We show by an example that the right-hand side of Equation (3.54) can have the wrong sign. We let η be the L^2-entropy $\eta(u) = u^2/2$ and consider (3.1a), (3.1b) with the periodic, smooth initial data

$$u^{0,\alpha}(x) = \sin(x), \quad x \in [0, 2\pi). \tag{3.55}$$

We approximate the solution to (3.1a), (3.1b) and (3.55) on the intervall $[0, 2\pi)$ at time $T = 2$ by a spectral scheme and plot at every point a discrete approximation of the quantity

$$\Delta E(x,t) = \alpha((\overline{u}^\alpha_t(x,t))^2 - 2\,(\overline{u}^\alpha_x(x,t))^3), \tag{3.56}$$

which corresponds to the entropy 'dissipation' in (3.54). In addition, we compute an approximation of the integral of (3.56) over the spatial domain $[0, 2\pi)$:

$$\Delta\overline{E}(t) = \alpha \int_{\mathbb{R}} (\overline{u}^\alpha_t(x,t))^2 - 2\,(\overline{u}^\alpha_x(x,t))^3)\,dx. \tag{3.57}$$

α	0.8	0.4	0.2	0.1	0.05
$\Delta\overline{E}(2)$	0.4094	0.6159	0.7864	0.9056	0.9897

Table 3.1: An approximation of the quantity $\Delta\overline{E}(2)$ for different α for the initial value problem (3.1a), (3.1b), (3.55).

We observe in the numerical experiments, that the quantity (3.57) (Table 3.1) and the peak of the approximation to the local entropy 'dissipation' (3.56) at $x = \pi$ (Figures 3.2 and 3.4) increase as $\alpha \to 0$. These quantities would be smaller or equal to zero if the entropy inequality were satisfied for this particular example.

As entropy solutions of the inviscid Burgers' equation satisfy the entropy inequality for any convex entropy function η, and hence also for L^p-entropies (3.50), this means that the function u obtained from u^α in the limit $\alpha \to 0$ is not necessarily the entropy solution, as we cannot show the entropy inequality for entropies of the form (3.50).

This leads us to the conclusion that the time dependent filter (3.1b) neither succeeds in regularizing Burgers' equation in such a way that the limit u of solutions u^α to (3.1) is the entropy solution of Burgers' equation.

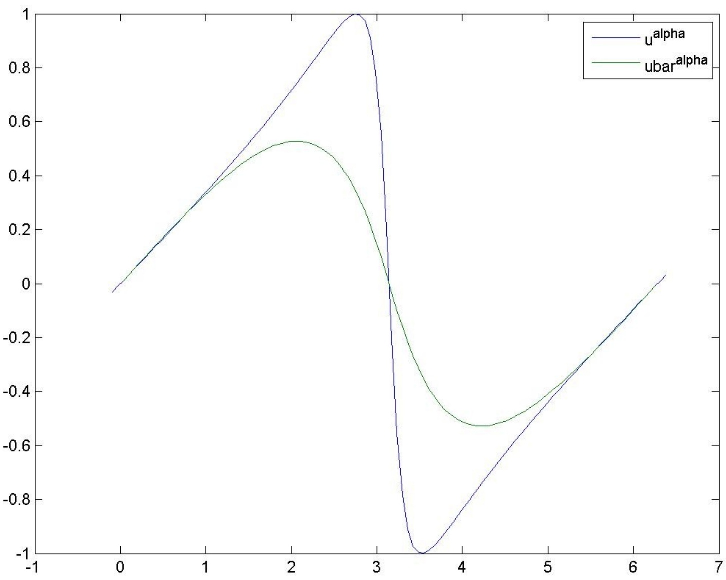

Figure 3.1: Numerical approximation of the initial value problem (3.1a), (3.1b), (3.55) at time $T = 2$. $\alpha = 0.5$, 100 meshpoints.

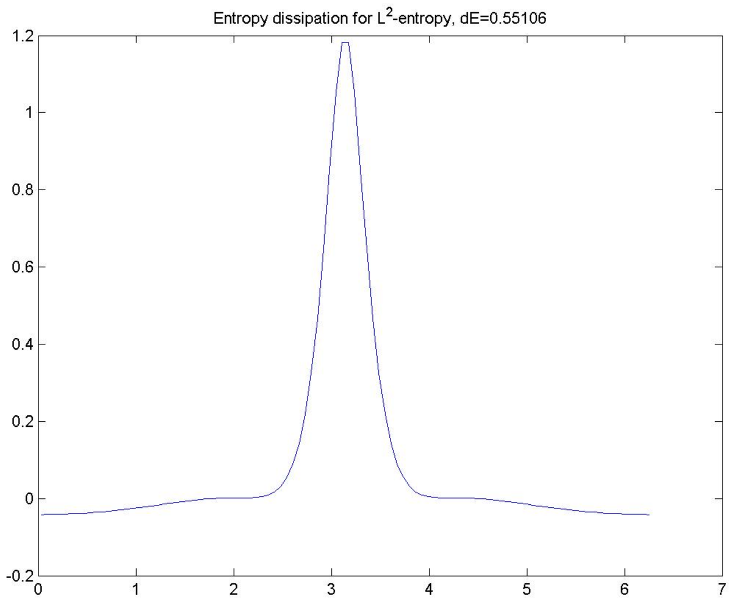

Figure 3.2: Numerical approximation of the entropy 'dissipation' (3.56) for the problem (3.1a), (3.1b), (3.55) at time $T = 2$. $\alpha = 0.5$, 100 meshpoints.

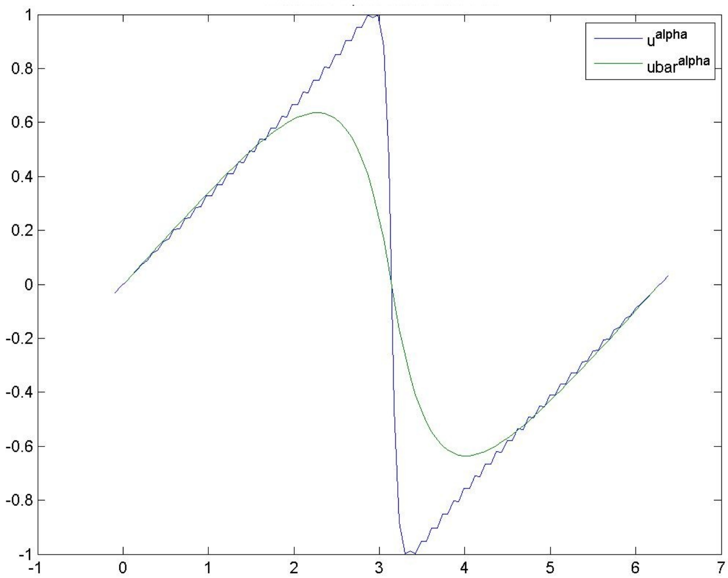

Figure 3.3: Numerical approximation of the initial value problem (3.1a), (3.1b), (3.55) at time $T = 2$. $\alpha = 0.2$, 100 meshpoints.

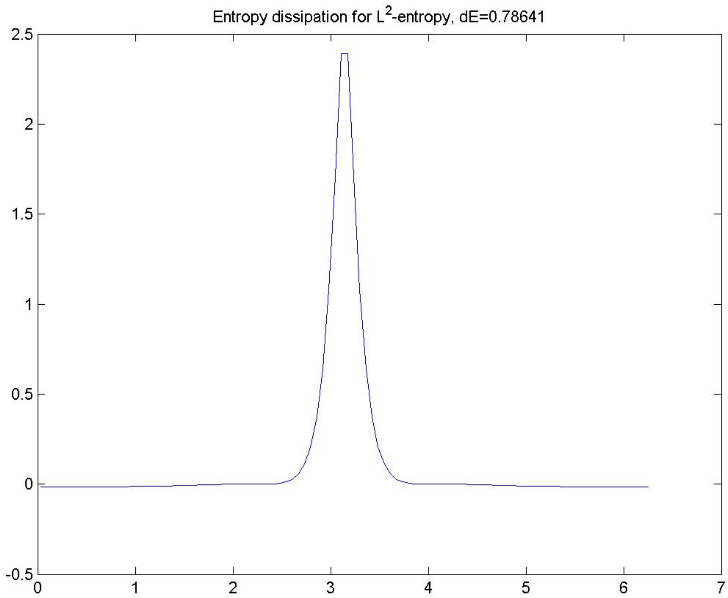

Figure 3.4: Numerical approximation of the entropy 'dissipation' (3.56) for the problem (3.1a), (3.1b), (3.55) at time $T = 2$. $\alpha = 0.2$, 100 meshpoints.

Chapter 4

Numerical experiments with the shallow water equations

Having investigated the effect of filtering the velocity in Burgers' equation, we could also ask whether convolving one or several variables with a mollifier is a justifiable technique for regularizing the solution of a system of conservation laws. Since existence and uniqueness of solutions of general systems of conservation laws is still an unsolved problem, we will only apply the method to the particular case of the 1d shallow water equations, where the properties of the solution are well known. These read in conservative form

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{h^2}{2} \end{pmatrix}_x = 0 \tag{4.1a}$$

$$\begin{pmatrix} h \\ q \end{pmatrix} (x,0) = \begin{pmatrix} h_0 \\ q_0 \end{pmatrix} (x), \tag{4.1b}$$

where h and $v := q/h$ denote the height/depth and the velocity of the fluid respectively (we set the gravity constant $g = 1$). We denote $\mathbf{u} := (h,q)^T$. By formally differentiating the flux function $f(\mathbf{u}) := (q, q^2/h + h^2/2)^T$, (4.1) can be rewritten as

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{h^2} + h & 2\frac{q}{h} \end{pmatrix} \begin{pmatrix} h \\ q \end{pmatrix}_x = 0. \tag{4.2}$$

A third way of writing these differential equations is in the physical variables h and v,

$$\begin{pmatrix} h \\ v \end{pmatrix}_t + \begin{pmatrix} v & h \\ 1 & v \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}_x = 0. \tag{4.3}$$

This suggests that there are various ways of applying the averaging technique, which we have examined for Burgers' equation; one could either convolve components of the flux in the conservative form (4.1) with a mollifier, or one or several components of the matrices in (4.2) and (4.3). In the following we are going to investigate several of these possibilities numerically. We will use the Helmholtz filter,

$$g_\alpha(x) := \frac{1}{2\alpha} e^{-\frac{|x|}{\alpha}}, \tag{4.4}$$

for our numerical experiments, since it has the advantage that we can write the convolution $\overline{u}^\alpha(x) = g_\alpha * u^\alpha(x)$ as a differential equation for u^α, that is,

$$u^\alpha = \overline{u}^\alpha - \alpha^2 \overline{u}^\alpha_{xx}. \tag{4.5}$$

For convenience, we will omit writing the dependencies of the solution quantities on α in the following.

4.1 Averaging the entries of the Jacobian matrix of the flux function

4.1.1 Averaging all entries of the Jacobian in (4.2)

In a first step we convolve all entries of the Jacobian matrix of the flux function in (4.2) with the mollifier g given in (4.4). This has no effect on the coefficients in the upper row of the matrix, since they are constant. We obtain

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ \overline{-\frac{q^2}{h^2} + h} & 2\overline{\left(\frac{q}{h}\right)} \end{pmatrix} \begin{pmatrix} h \\ q \end{pmatrix}_x = 0. \tag{4.6}$$

where we have denoted

$$\overline{\left(-\frac{q^2}{h^2} + h\right)}(x) := g_\alpha * \left(-\frac{q^2}{h^2} + h\right)(x)$$

and

$$\overline{\left(\frac{q}{h}\right)}(x) := g_\alpha * \left(\frac{q}{h}\right)(x).$$

To get a numerical scheme, we discretize the spatial domain by an equidistant grid with gridpoints denoted by $x_{j+\frac{1}{2}}$, $j \in \mathbb{N}$. The cell midpoints are denoted by x_j, $j \in \mathbb{N}$ and $\Delta x = x_{j+1} - x_j$. In this and in the following sections, the approximations of $h(x_j, t)$ and $q(x_j, t)$ are denoted by $h_j(t)$, $q_j(t)$

respectively and $\mathbf{u}_j(t) := (h_j(t), q_j(t))^T$. Moreover, we let $v_j(t) := q_j(t)/h_j(t)$. The spatial derivatives are approximated by central differences in the case of Neumann boundary conditions and by a (pseudo) spectral derivative in the case of periodic boundary conditions. Furthermore, we will denote from now on the vectors of approximated quantities by $\underline{y}(t) := (\dots, y_j(t), \dots)^T$, j ranging from 1 to N in the periodic case and from 0 to $N+1$ in the case of Neumann boundary condtions. Specifically, in this section, we will need,

$$\underline{c}(t) = (\dots, c_j(t), \dots)^T := \left(\dots, -\frac{q_j^2(t)}{h_j^2(t)} + h_j(t), \dots\right)^T,$$

and

$$\underline{d}(t) = (\dots, d_j(t), \dots)^T := \left(\dots, 2\frac{q_j(t)}{h_j(t)}, \dots\right)^T.$$

The semidiscrete scheme then reads,

$$\frac{d}{dt}\begin{pmatrix} h_j \\ q_j \end{pmatrix}(t) = -\begin{pmatrix} 0 & 1 \\ \overline{c_j}(t) & \overline{d_j}(t) \end{pmatrix}\begin{pmatrix} D_0 h_j \\ D_0 q_j \end{pmatrix}(t) := \mathcal{L}(\mathbf{u})_j(t), \tag{4.7a}$$

$$\overline{c_j}(t) = \left((\mathbb{I} - \alpha^2 D^2)^{-1}\underline{c}(t)\right)_j, \tag{4.7b}$$

$$\overline{d_j}(t) = \left((\mathbb{I} - \alpha^2 D^2)^{-1}\underline{d}(t)\right)_j, \tag{4.7c}$$

$$j = 1, \dots, N,$$

where we denoted by $D_0 h_j$, $D_0 q_j$ and D^2 the approximations of the first and second derivatives. For Neumann boundary conditions these are the central differences

$$D_0 h_j(t) := \frac{1}{2\Delta x}(h_{j+1} - h_{j-1})(t), \quad D_0 q_j(t) := \frac{1}{2\Delta x}(q_{j+1} - q_{j-1})(t), \tag{4.8}$$

and D^2 is the matrix of second differences,

$$D^2 = \frac{1}{\Delta x^2}\begin{pmatrix} -1 & 1 & 0 & \dots & & 0 \\ 1 & -2 & 1 & \ddots & & \vdots \\ 0 & 1 & -2 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & 0 & 1 & -2 & 1 \\ 0 & \dots & & 0 & 1 & -1 \end{pmatrix}. \tag{4.9}$$

For the periodic boundary conditions, we use

$$D_0 h_j(t) := \left(D^1 \underline{h}(t)\right)_j, \quad D_0 q_j(t) := \left(D^1 \underline{q}(t)\right)_j, \tag{4.10}$$

where

$$D^1 = \frac{1}{\Delta x}\begin{pmatrix} 0 & & & -\frac{1}{2}\cot\frac{1\Delta x}{2} \\ -\frac{1}{2}\cot\frac{1\Delta x}{2} & \ddots & \ddots & \frac{1}{2}\cot\frac{2\Delta x}{2} \\ \frac{1}{2}\cot\frac{2\Delta x}{2} & \ddots & & -\frac{1}{2}\cot\frac{3\Delta x}{2} \\ -\frac{1}{2}\cot\frac{3\Delta x}{2} & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{2}\cot\frac{1\Delta x}{2} \\ \frac{1}{2}\cot\frac{1\Delta x}{2} & & & 0 \end{pmatrix}, \tag{4.11}$$

is the matrix corresponding to the spectral differentiation [24], to approximate the first derivative. To approximate the second derivative, we need the matrix

$$D^2 = \frac{1}{\Delta x^2}\begin{pmatrix} \ddots & \vdots & \\ \ddots & -\frac{1}{2}\csc^2\left(\frac{2\Delta x}{2}\right) & \\ \ddots & \frac{1}{2}\csc^2\left(\frac{1\Delta x}{2}\right) & \\ & -\frac{\pi^2}{3\Delta x} - \frac{1}{6} & \\ & \frac{1}{2}\csc^2\left(\frac{1\Delta x}{2}\right) & \ddots \\ & -\frac{1}{2}\csc^2\left(\frac{2\Delta x}{2}\right) & \ddots \\ & \vdots & \ddots \end{pmatrix}, \tag{4.12}$$

see [24].

Central differences or a spectral approximation of the derivative seem to be bad choices for a conservation law, but on the one hand, we are faced with a system written in non-conservative form which makes it impossible to use a scheme in flux form. On the other hand, we do not want to add too much numerical diffusion since we are mainly interested in the regularizing effect of averaging one or several variables and the diffusion could alter the convergence or non-convergence behavior. Moreover, we want to know whether averaging succeeds in stabilizing the numerical scheme similarly to how a numerical diffusion operator does.

For the discretization in time we denote the time steps by t^n, $0 \le n \le N_T$, with $t^{N_T} = T$ and the approximated solutions at (x_j, t^n) by $\mathbf{u}_j^n$, $j = 1, \ldots, N$, $n = 0, \ldots, N_T$. We choose $\Delta t^n := t^n - t^{n-1}$ small enough such that it satisfies in each step an appropriate CFL-condition, here

$$\frac{\Delta t^{n+1}}{\Delta x} \max_{i,j} |\lambda_i(\mathbf{u}_j^n)| \le \frac{1}{2},$$

where $\lambda_i(\mathbf{u}_j^n)$ $i = 1,2$, denote the eigenvalues of the matrix in (4.7a) at time t^n. In the experiments, we will use Forward Euler or the second order SSP-Runge-Kutta method,

$$\begin{aligned} \mathbf{u}^* &= \mathbf{u}^n + \Delta t^{n+1} \mathcal{L}(\mathbf{u}^n) \\ \mathbf{u}^{**} &= \mathbf{u}^* + \Delta t^{n+1} \mathcal{L}(\mathbf{u}^*) \\ \mathbf{u}^{n+1} &= \frac{1}{2}(\mathbf{u}^n + \mathbf{u}^{**}) \end{aligned}$$

where we denoted $\mathbf{u}^n := (\ldots, \mathbf{u}_j^n, \ldots)^T$, and $\mathcal{L}(\mathbf{u}^n) := (\ldots, \mathcal{L}(\mathbf{u})_j(t^n), \ldots)^T$ for the timestepping. For the spectral scheme, we could use a higher order timestepping method to gain further accuracy as long as the solution is sufficiently smooth (which is usually not the case). We test the above scheme for different α and Δx with the following initial data:

1. *Periodic, smooth initial data*

$$q_0(x) = \sin(x), \tag{4.13a}$$
$$h_0(x) = 1, \qquad x \in [0, 2\pi), \tag{4.13b}$$

2. *Riemann Problem 1 with $q_L > q_R$ and Neumann boundary conditions*

$$q_0(x) = \begin{cases} 1 & \text{if } x \in [-2, 0), \\ -1 & \text{if } x \in [0, 2], \end{cases} \tag{4.14a}$$
$$h_0(x) = 1, \qquad x \in [-2, 2], \tag{4.14b}$$

3. *Riemann Problem 2 with $q_L < q_R$ and Neumann boundary conditions*

$$q_0(x) = \begin{cases} -0.5 & \text{if } x \in [-3, 0), \\ 0.5 & \text{if } x \in [0, 3], \end{cases} \tag{4.15a}$$
$$h_0(x) = 1, \qquad x \in [-3, 3]. \tag{4.15b}$$

4. *Dam break problem with $h_L < h_R$ and Neumann boundary conditions*

$$q_0(x) = 0, \qquad x \in [-3, 3], \tag{4.16a}$$
$$h_0(x) = \begin{cases} 4 & \text{if } x \in [-3, 0), \\ 0.25 & \text{if } x \in [0, 3], \end{cases} \tag{4.16b}$$

The exact solutions of the three Riemann problems are for (4.14)

$$q(x, T) = \begin{cases} 1 & \text{if } x < -0.8546\,T, \\ 0 & \text{if } -0.8546\,T \leq x < 0.8546\,T, \\ -1 & \text{if } x \geq 0.8546\,T, \end{cases}$$

$$h(x, T) = \begin{cases} 1 & \text{if } x < -0.8546\,T, \\ 2.1701 & \text{if } -0.8546\,T \leq x < 0.8546\,T, \\ 1 & \text{if } x \geq 0.8546\,T, \end{cases}$$

for (4.15)

$$q(x,T) = \begin{cases} -0.5 & \text{if } x < -1.5\,T, \\ \frac{1}{27}(\frac{3}{2} + 2\frac{x}{T})(\frac{3}{2} + \frac{x}{T})^2 & \text{if } -1.5\,T \le x < -0.75\,T, \\ 0 & \text{if } -0.75\,T \le x < 0.75\,T, \\ \frac{1}{27}(-\frac{3}{2} + 2\frac{x}{T})(\frac{3}{2} + \frac{x}{T})^2 & \text{if } 0.75\,T \le x < 1.5\,T, \\ 0.5 & \text{if } x \ge 1.5\,T, \end{cases}$$

$$h(x,T) = \begin{cases} 1 & \text{if } x < -1.5\,T, \\ \frac{1}{9}(\frac{3}{2} + \frac{x}{T})^2 & \text{if } -1.5\,T \le x < -0.75\,T, \\ 0.5625 & \text{if } -0.75\,T \le x < 0.75\,T, \\ \frac{1}{9}(\frac{3}{2} + \frac{x}{T})^2 & \text{if } 0.75\,T \le x < 1.5\,T, \\ 1 & \text{if } x \ge 1.5\,T, \end{cases}$$

and for (4.16)

$$q(x,T) = \begin{cases} 0 & \text{if } x < -2\,T, \\ \frac{1}{27}(4 + 2\frac{x}{T})(4 - \frac{x}{T})^2 & \text{if } -2\,T \le x < 0.545\,T, \\ 2.2503 & \text{if } 0.545\,T \le x < 2.0821\,T, \\ 0 & \text{if } x \ge 2.0821\,T, \end{cases}$$

$$h(x,T) = \begin{cases} 4 & \text{if } x < -2\,T, \\ \frac{1}{9}(4 - \frac{x}{T})^2 & \text{if } -2\,T \le x < 0.545\,T, \\ 1.32633 & \text{if } 0.545\,T \le x < 2.0821\,T, \\ 0.25 & \text{if } x \ge 2.0821\,T. \end{cases}$$

We found the exact solutions by computing the rarefaction curves and the Hugoniot locus (shock curves) emanating from the left and the right states of the Riemann problems respectively; and their intersections, which yield the middle states (see e.g. [12, 22, 23]). For the sine wave initial data we will use an approximation computed with a Finite Volume scheme with Roe flux on a mesh with 2000 points as a reference solution. If not otherwise mentionned, we will compute an approximation of the solutions of these four initial value problems at time $T = 1$.

In Figures 4.1 and 4.2, we observe that the spectral scheme yields a good approximation of the periodic initial data as long as the solution stays smooth. At time $T = 1.4$ shocks have developed in the solution and the approximation becomes oscillatory and eventually blows up for larger times. We compute an approximation of the L^1-norm of the difference between the Roe approximation and the approximation with the spectral scheme (4.7) by

$$\mathcal{E}^{\Delta x}_{\text{Roe}} = \Delta x \sum_{j=1}^{N} \Big(|h_j^T - h^{\text{Roe}}(x_j,T)| + |q_j^T - q^{\text{Roe}}(x_j,T)| \Big),$$

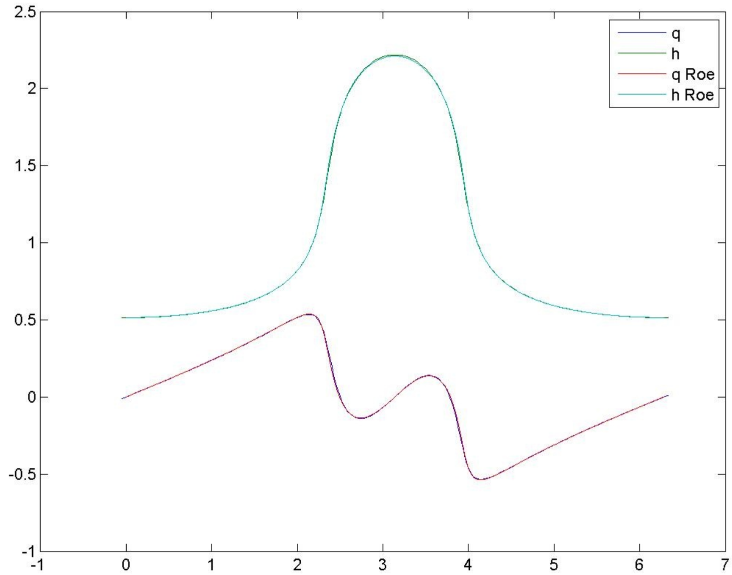

Figure 4.1: Numerical approximation of the initial value problem (4.13) at time $T = 1$ by Scheme (4.7). $\alpha = 0.05$, 200 meshpoints.

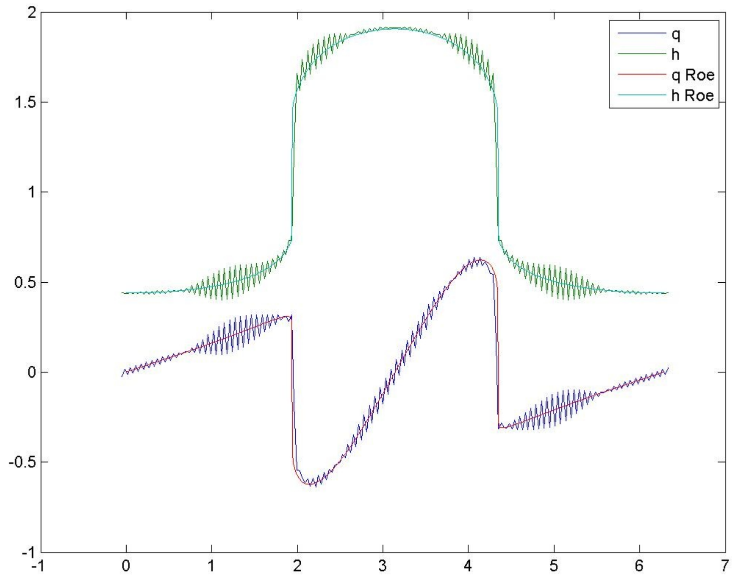

Figure 4.2: Numerical approximation of the initial value problem (4.13) at time $T = 1.4$ by Scheme (4.7). $\alpha = 0.05$, 200 meshpoints.

h_j^T, q_j^T respectively, denoting the approximations of (4.7) at time T and $h^{\text{Roe}}(x_j,T)$, $q^{\text{Roe}}(x_j,T)$ denoting the approximations with the Roe scheme. We obtain $\mathcal{E}_{\text{Roe}}^{\Delta x} = 0.0309$ at time $T = 1$ and $\mathcal{E}_{\text{Roe}}^{\Delta x} = 0.5102$ at time $T = 1.4$ for $N = 200$ meshpoints.

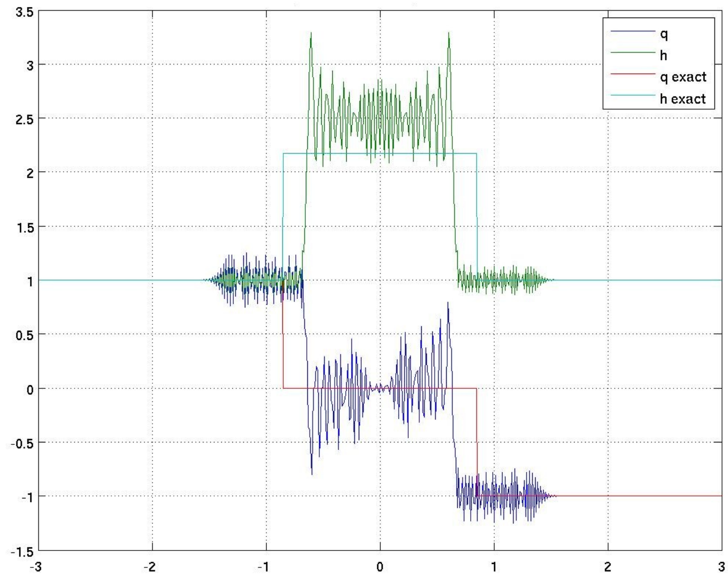

Figure 4.3: Numerical approximation of Riemann Problem 1, (4.14), at time $T = 1$ by Scheme (4.7). $\alpha = 0.5$, 500 meshpoints.

Since we observe a lot of oscillations in the approximations to the Riemann problems 1, 2 and 3, as it can be seen in Figure 4.3 for the Riemann problem (4.14) (we have not included figures for the other two Riemann problems), and we have seen that at least in the case of the regularized Burgers' equation, smoothing the initial data results in smooth solutions for all times, we try the same approach for this system in the hope of decreasing the oscillations. We introduce a parameter $\delta > 0$ and use instead of the Riemann problem 1 the following initial data:

$$q_0(x) = \begin{cases} 1 & \text{if } x \in [-2, 0), \\ 2\exp\{1 - \frac{1}{1-(\frac{x}{\delta})^2}\} - 1 & \text{if } x \in [0, \delta), \\ -1 & \text{if } x \in [\delta, 2], \end{cases} \tag{4.17a}$$

$$h_0(x) = 1, \qquad x \in [-2, 2]. \tag{4.17b}$$

Instead of Riemann Problem 2, we test with the smoothened initial data

$$q_0(x) = \begin{cases} -0.5 & \text{if } x \in [-3, -\delta], \\ \exp\{1 - \frac{1}{1-(\frac{x}{\delta})^2}\} - 0.5 & \text{if } x \in (-\delta, 0], \\ 0.5 & \text{if } x \in (0, 3], \end{cases} \tag{4.18a}$$

$$h_0(x) = 1, \qquad x \in [-3, 3]. \tag{4.18b}$$

and Riemann Problem 3 is replaced by

$$q_0(x) = 0, \qquad x \in [-3, 3] \tag{4.19a}$$

$$h_0(x) = \begin{cases} 4 & \text{if } x \in [-3, 0], \\ 3.75\,\exp\{1 - \frac{1}{1-(\frac{x}{\delta})^2}\} + 0.25 & \text{if } x \in (0, \delta), \\ 0.25 & \text{if } x \in [\delta, 3]. \end{cases} \tag{4.19b}$$

We compare the approximations to (4.14) in Figure 4.3 with those to Problem (4.17) in Figure 4.4 and see that the oscillations have decreased considerably. The same behavior can be observed in the approximations to the second Riemann problem, (4.15) and (4.18) (we have not included the corresponding plots here): Introducing the parameter δ decreases the oscillations considerably. Moreover, it does not seem to influence the convergence behavior of the numerical schemes tested here. Therefore we will conduct the experiments of this and the following sections with the smoothened Riemann initial data (4.17), (4.18) and (4.19).

Convergence analysis

Comparing the approximations of (4.17) with (4.7) in Figures 4.5 and 4.6 to the exact solution of the Riemann problem, we observe that the approximations give us a wrong middle state in the variable h. Moreover, the shock speeds are wrong. As α decreases, we observe more oscillations, so averaging the entries of the Jacobian indeed regularizes the solution to (4.6).

As we can see in Figures 4.7 and 4.8, the approximations of Problem (4.18) with Scheme (4.7) seem to converge to the correct functions. We do not observe a lot of oscillations, even for small α. Apparently, larger α result in steeper gradients in the two regions where we observe a rarefaction wave in the exact solution. We calculate an approximation of the error in the L^1-norm:

$$\mathcal{E}^{\Delta x} := \Delta x \sum_{j=1}^{N} \Big(|h(x_j, T) - h_j^T| + |q(x_j, T) - q_j^T| \Big),$$

where h_j^T and q_j^T are the approximations to h, q respectively, at (x_j, T).

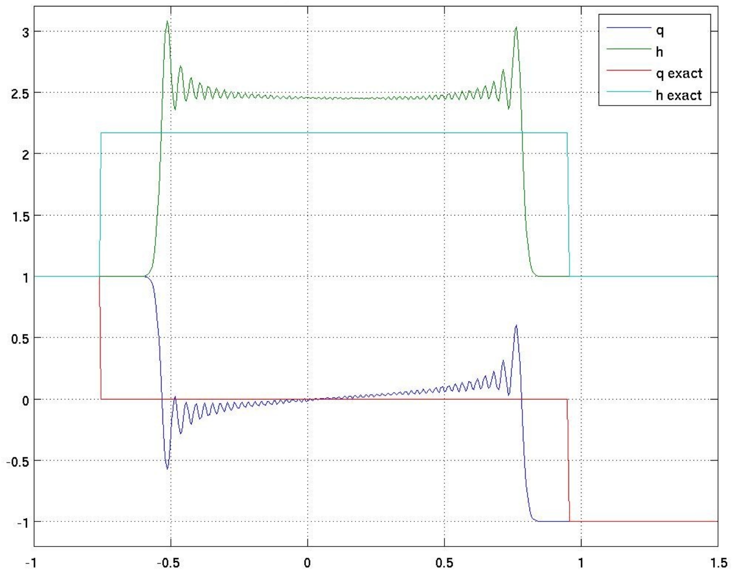

Figure 4.4: Numerical approximation of the initial value problem (4.17) with $\delta = 0.2$ at time $T = 1$ by Scheme (4.7). $\alpha = 0.4$, 400 meshpoints.

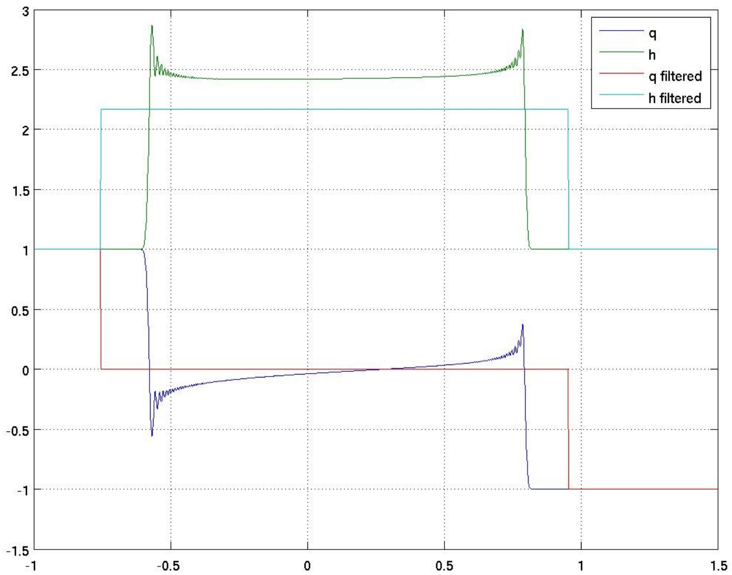

Figure 4.5: Numerical approximation of the initial value problem (4.17) with $\delta = 0.2$ at time $T = 1$ by Scheme (4.7). $\alpha = 0.2$, 1600 meshpoints.

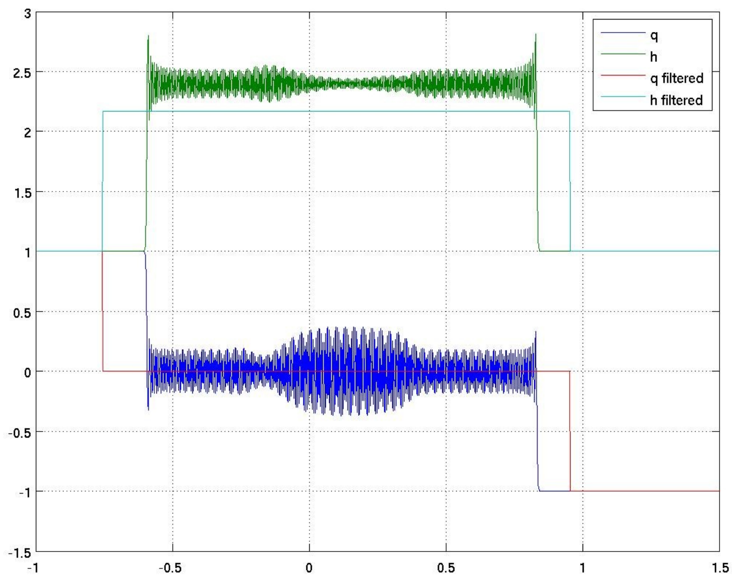

Figure 4.6: Numerical approximation of the initial value problem (4.17) with $\delta = 0.2$ at time $T = 1$ by Scheme (4.7). $\alpha = 0.05$, 1600 meshpoints.

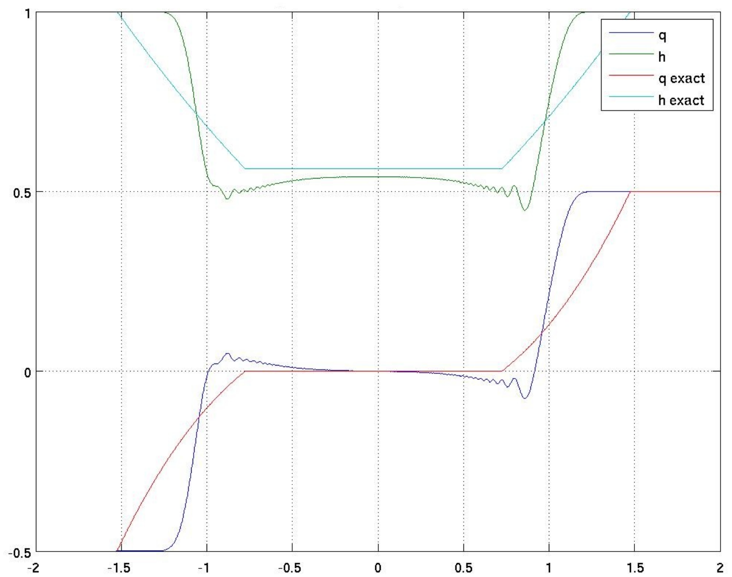

Figure 4.7: Numerical approximation of the initial value problem (4.18) with $\delta = 0.05$ at time $T = 1$ by Scheme (4.7). $\alpha = 0.2$, 1600 meshpoints, $\mathcal{E}^{\Delta x} = 0.4167$.

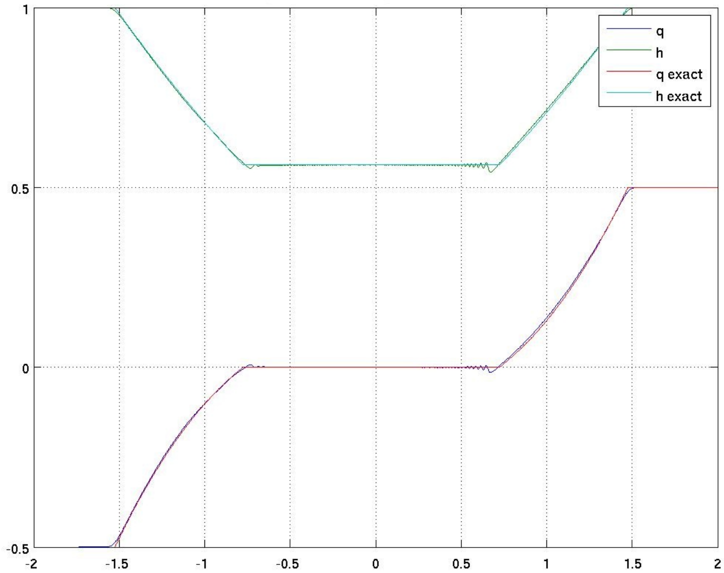

Figure 4.8: Numerical approximation of the initial value problem (4.18) with $\delta = 0.05$ at time $T = 1$ by Scheme (4.7) using a Forward Euler timestepping. $\alpha = 0.0125$, 1600 meshpoints, $\mathcal{E}^{\Delta x} = 0.0178$.

Regarding the dam break Riemann problem 3, we see in Figures 4.9 and 4.10 that the approximations differ considerably from the exact solution. Again, decreasing α results in an increase in oscillations. We conclude that Scheme (4.7) gives satisfactory results as long as the solution to (4.1) remains smooth, but fails to converge if shocks are to be expected in the solution.

4.1.2 Averaging the velocity v in (4.2)

In a second try, we filter only the velocity $v := q/h$ in the Jacobian of the flux function in (4.2). This yields,

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ -\overline{\left(\frac{q}{h}\right)}^2 + h & 2\overline{\left(\frac{q}{h}\right)} \end{pmatrix} \begin{pmatrix} h \\ q \end{pmatrix}_x = 0. \tag{4.20}$$

where we denoted

$$\overline{\left(\frac{q}{h}\right)}(x) := g_\alpha * \left(\frac{q}{h}\right)(x).$$

Using $v := q/h$ and $\overline{v} = \overline{q/h}$, we have by (4.5)

$$v = \overline{v} - \alpha^2\,\overline{v}_{xx},$$

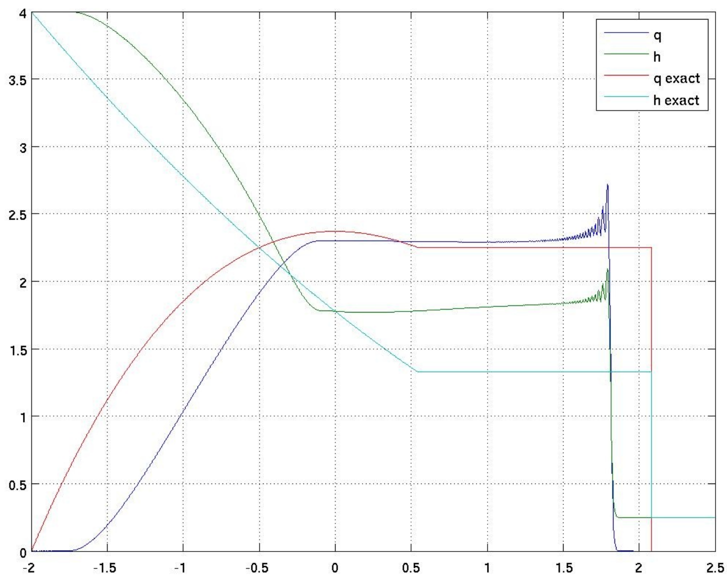

Figure 4.9: Numerical approximation of the initial value problem (4.19) with $\delta = 0.5$ at time $T = 1$ by Scheme (4.7). $\alpha = 0.4$, 1600 meshpoints.

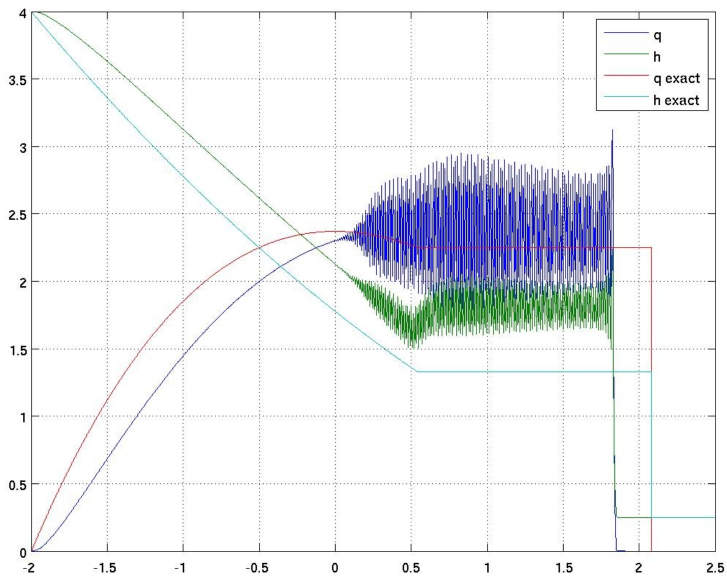

Figure 4.10: Numerical approximation of the initial value problem (4.19) with $\delta = 0.5$ at time $T = 1$ by Scheme (4.7). $\alpha = 0.05$, 1600 meshpoints.

and can thus rewrite (4.20) as a system of differential equations for h and $\overline{v}$,

$$\begin{aligned} h_t + (h\overline{v})_x - \alpha^2 (h\,\overline{v}_{xx})_x &= 0, \\ (h\overline{v})_t - \alpha^2 (h\,\overline{v}_{xx})_t + (\overline{v}^2\,h)_x + \left(\frac{h^2}{2}\right)_x - 2\,\alpha^2\,\overline{v}(h\,\overline{v}_{xx})_x &= 0. \end{aligned}$$

This system is not conservative in contrast to the shallow water system. As before, we discretize in space and denote the approximations to h, q and v by h_j, q_j and v_j respectively. Again, we use central differences to approximate the spatial derivatives in the case of Neumann boundary conditions and a spectral approximation of the derivatives in the case of periodic boundary conditions and smooth initial data. We arrive at the semidiscrete formulation of the scheme,

$$\frac{d}{dt}\begin{pmatrix} h_j \\ q_j \end{pmatrix}(t) = -\begin{pmatrix} 0 & 1 \\ \overline{c_j}(t) & \overline{d_j}(t) \end{pmatrix}\begin{pmatrix} D_0 h_j \\ D_0 q_j \end{pmatrix}(t) := \mathcal{L}(\mathbf{u})_j(t), \tag{4.21a}$$

$$\overline{v_j}(t) = \left((\mathbb{I} - \alpha^2 D^2)^{-1}\underline{v}(t)\right)_j, \tag{4.21b}$$

$$\overline{c_j}(t) = (-\overline{v_j}^2 + h_j)(t), \tag{4.21c}$$

$$\overline{d_j}(t) = 2\,\overline{v_j}(t), \tag{4.21d}$$

$$j = 1, \ldots, N,$$

where $D_0 h_j$, $D_0 q_j$ and D^2 have been defined in (4.8), (4.11), (4.9) and (4.12). We test the scheme with the initial data (4.13), (4.17), (4.18) and (4.19). We have also tested the scheme on the Riemann problems (4.14), (4.15) and (4.16), but we do not include the results here since the approximated solutions to these initial conditions are even more oscillatory, as we have seen it for Scheme (4.7) and we do not obtain convergence to the exact solutions either.

Similarly to the first scheme considered, we observe a good approximation of the periodic initial data (Figure 4.11) as long as the solution stays smooth. After shocks have developed in the exact solution, oscillations appear in the approximation and it finally blows up (no figure).

In Figures 4.12, 4.13 and 4.14 we see the approximations to the smoothened Riemann problems 1, 2 and 3. We observe that the rarefaction waves of Riemann Problem 2 are quite well resolved, whereas the scheme gives us approximations with wrong shock speeds and middle states for the Riemann problems 1 and 3. In contrast to Scheme (4.7) these approximations are more oscillatory too which we could have expected since we have averaged only the variable v.

In a similar way, we did numerical experiments for the equations,

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ -\overline{\left(\frac{q}{h}\right)}^2 + \overline{h} & 2\overline{\left(\frac{q}{h}\right)} \end{pmatrix}\begin{pmatrix} h \\ q \end{pmatrix}_x = 0, \tag{4.22}$$

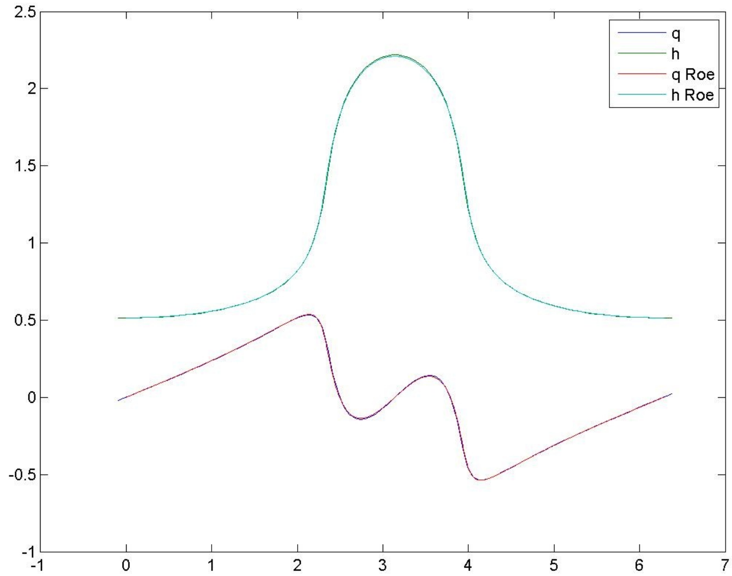

Figure 4.11: Numerical approximation of the initial value problem (4.13) at time $T = 1$ by Scheme (4.21). $\alpha = 0.05$, 100 meshpoints, $\mathcal{E}^{\Delta x}_{\text{Roe}} = 0.0368$.

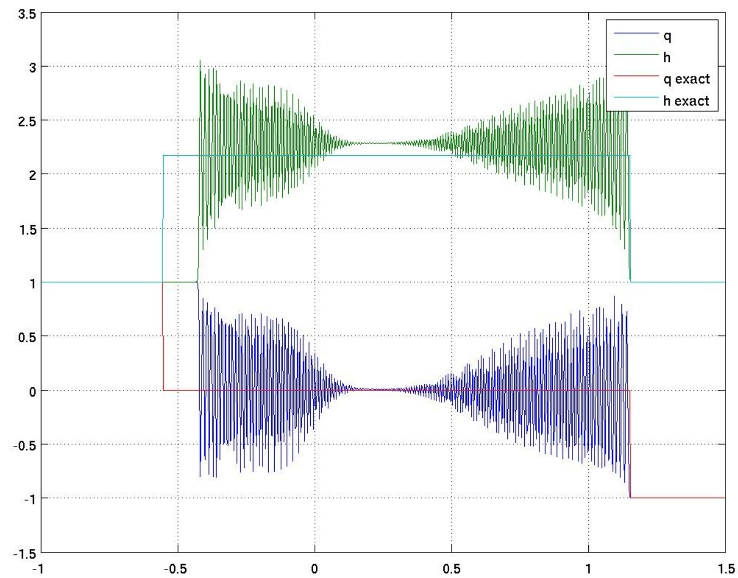

Figure 4.12: Numerical approximation of the initial value problem (4.17) with $\delta = 0.6$ at time $T = 1$ by Scheme (4.21). $\alpha = 0.2$, 800 meshpoints.

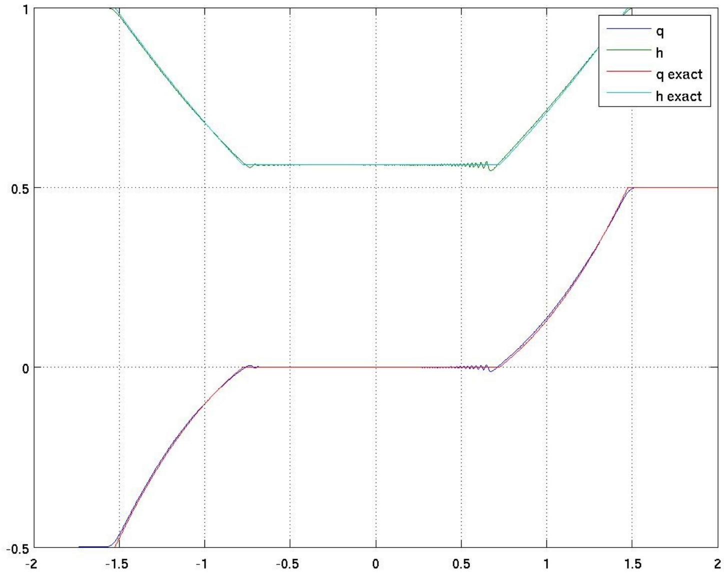

Figure 4.13: Numerical approximation of the initial value problem (4.18) with $\delta = 0.05$ at time $T = 1$ by Scheme (4.21). $\alpha = 0.0125$, 1600 meshpoints, $\mathcal{E}^{\Delta x} = 0.0186$.

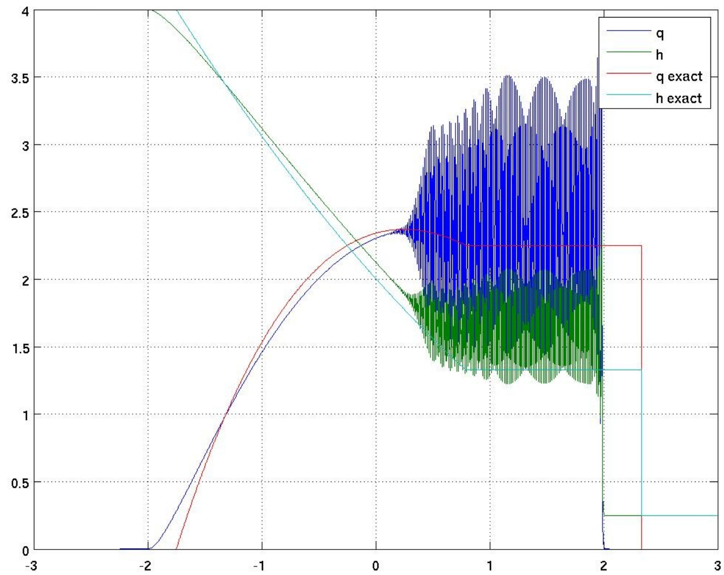

Figure 4.14: Numerical approximation of the initial value problem (4.19) with $\delta = 0.5$ at time $T = 1$ by Scheme (4.21). $\alpha = 0.05$, 1600 meshpoints.

where the variables v and h are filtered. The approximations of the smoothened Riemann problems (4.17) and (4.19) did not converge to the exact solution either. We have not included the plots.

4.2 Filtering the entries of the matrix in (4.3)

4.2.1 Filtering all entries of the matrix

In a next step, we investigate the effect of filtering entries of the matrix in the equation for v and h, (4.3). We start by convolving all entries with g_α. This yields

$$\begin{pmatrix} h \\ v \end{pmatrix}_t + \begin{pmatrix} \overline{v} & \overline{h} \\ 1 & \overline{v} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}_x = 0. \tag{4.23}$$

with

$$h = \overline{h} - \alpha^2\, \overline{h}_x x \quad \text{and} \quad v = \overline{v} - \alpha^2\, \overline{v}_{xx}.$$

We can rewrite (4.23) as a system of equations for $\overline{v}$ and $\overline{h}$,

$$\overline{h}_t + (\overline{v}\overline{h})_x - \alpha^2(\overline{h}_{xxt} + \overline{v}\overline{h}_{xxx} + \overline{h}\overline{v}_{xxx}) = 0,$$

$$(\overline{v}\overline{h})_t + \left(\frac{\overline{h}^2}{2}\right)_x + (\overline{h}\overline{v}^2)_x - \alpha^2(\overline{v}\overline{h}_{xxt} + \overline{h}\overline{v}_{xxt} + \overline{h}\overline{h}_{xxx} + 2\,\overline{h}\overline{v}\overline{v}_{xxx} + \overline{v}^2\overline{h}_{xxx}) = 0.$$

To obtain a numerical scheme, we approximate the spatial derivatives by central differences in the case of Neumann boundary conditions and by spectral differences in the case of periodic boundary conditions, as before. We arrive at the following semidiscrete formulation

$$\frac{d}{dt}\begin{pmatrix} h_j \\ v_j \end{pmatrix}(t) = -\begin{pmatrix} \overline{v_j}(t) & \overline{h_j}(t) \\ 1 & \overline{v_j}(t) \end{pmatrix} \begin{pmatrix} D_0 h_j \\ D_0 v_j \end{pmatrix}(t) := \mathcal{L}(\mathbf{u})_j(t), \tag{4.24a}$$

$$\overline{v_j}(t) = \left((\mathbb{I} - \alpha^2 D^2)^{-1}\underline{v}(t)\right)_j, \tag{4.24b}$$

$$\overline{h_j}(t) = \left((\mathbb{I} - \alpha^2 D^2)^{-1}\underline{h}(t)\right)_j, \tag{4.24c}$$

$$j = 1, \ldots, N,$$

where $D_0 h_j$, $D_0 v_j$ and D^2 have been defined in (4.8), (4.11), (4.9) and (4.12). We test the scheme with the initial data (4.13), (4.17), (4.18) and (4.19).

Exactly as in the first two models considered, we observe wrong shock speeds and middle states in the approximations to the smoothened Riemann problems 1 and 3 (Figures 4.15 and 4.17) and a good approximation of the periodic problem (no figure) and Riemann Problem 2 (Figure 4.16). The

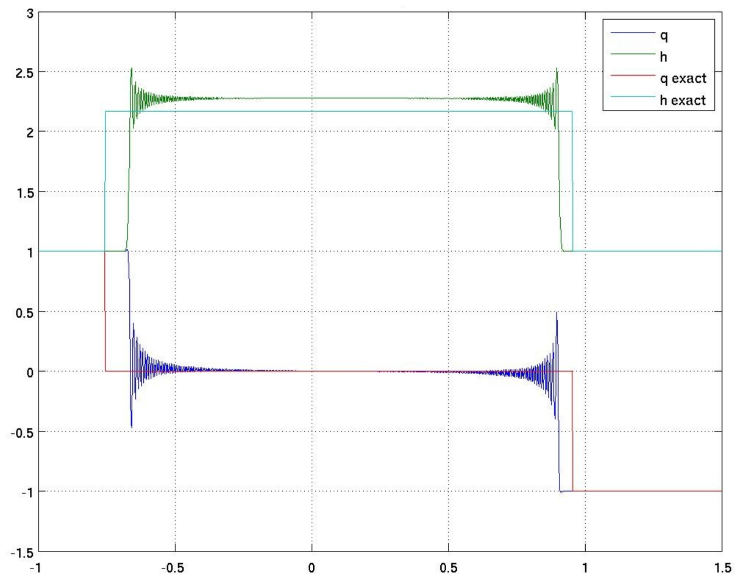

Figure 4.15: Numerical approximation of the initial value problem (4.17) with $\delta = 0.2$ at time $T = 1$ by Scheme (4.24). $\alpha = 0.2$, 1600 meshpoints.

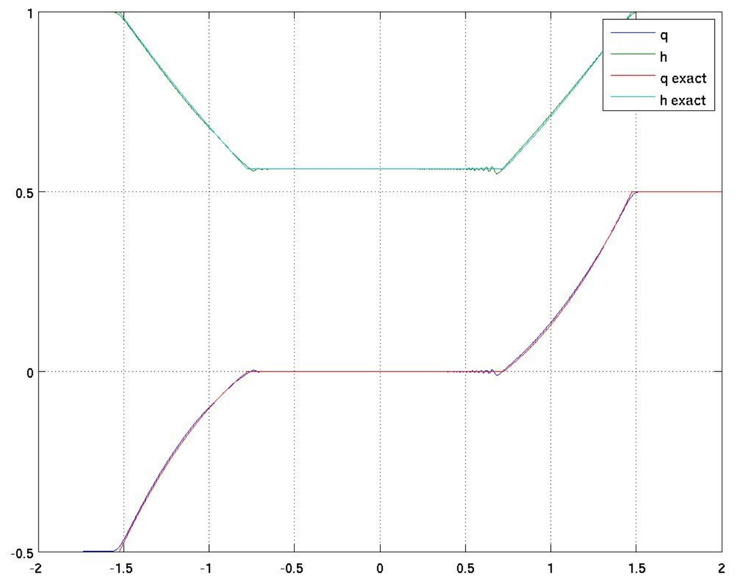

Figure 4.16: Numerical approximation of the initial value problem (4.18) with $\delta = 0.05$ at time $T = 1$ by Scheme (4.24). $\alpha = 0.0125$, 1600 meshpoints, $\mathcal{E}^{\Delta x} = 0.0162$.

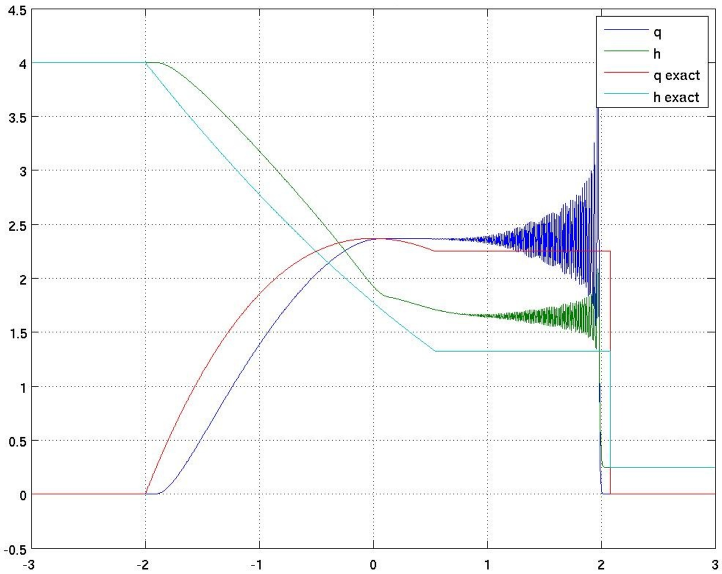

Figure 4.17: Numerical approximation of the initial value problem (4.19) with $\delta = 0.5$ at time $T = 1$ by Scheme (4.24). $\alpha = 0.2$, 1600 meshpoints.

behavior of the approximation to Riemann Problem 2 for large α is similar as for the first two problems, the gradients in the rarefaction regions of the exact solution become steeper for increasing α. In Riemann Problems 1 and 3 the oscillations grow larger as α decreases.

4.2.2 Filtering of the velocity v in (4.3)

As we did it for the non-conservative system (4.2), we now consider the variant of the model (4.23) with only v filtered,

$$\begin{pmatrix} h \\ v \end{pmatrix}_t + \begin{pmatrix} \overline{v} & h \\ 1 & \overline{v} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}_x = 0. \tag{4.25}$$

with

$$v = \overline{v} - \alpha^2 \, \overline{v}_{xx}.$$

Writing the equations in terms of the variables $\overline{v}$ and h, we get

$$\begin{aligned} h_t + (\overline{v}h)_x - \alpha^2 h \overline{v}_{xxx} &= 0, \\ (\overline{v}h)_t + \left(\frac{h^2}{2}\right)_x + (\overline{v}^2 \, h)_x - \alpha^2 (h\overline{v}_{xxt} + 2\, h\overline{v}\overline{v}_{xxx}) &= 0. \end{aligned}$$

This regularization has already been examined in [3] in a more general form, namely for the isentropic Euler equations,

$$\rho_t + \overline{v}\rho_x + \rho v_x = 0, \tag{4.26a}$$

$$v_t + \overline{v} v_x + \frac{p_x}{\rho} = 0, \tag{4.26b}$$

$$p = \kappa \rho^\gamma, \tag{4.26c}$$

$$\overline{v} - \alpha^2 \overline{v}_{xx} = v, \tag{4.26d}$$

where $\gamma > 0$ and $\kappa > 0$ are constants. Setting $\kappa = 1/2$ and $\gamma = 2$, this reduces to (4.23). In [3] the existence of smooth solutions for (4.26) is analyzed and they conclude that for $\gamma \neq 0$ discontinuities develop in finite time in the modified equations. Hence also in our case. We will perform numerical experiments to investigate whether system (4.25) approximates the solution of the shallow water equations.

In the same way as before, we approximate these equations using central differences or a spectral approximation of the spatial derivatives,

$$\frac{d}{dt}\begin{pmatrix} h_j \\ v_j \end{pmatrix}(t) = -\begin{pmatrix} \overline{v_j}(t) & h_j(t) \\ 1 & \overline{v_j}(t) \end{pmatrix}\begin{pmatrix} D_0 h_j \\ D_0 v_j \end{pmatrix}(t) := \mathcal{L}(\mathbf{u})_j(t), \tag{4.27a}$$

$$\overline{v_j}(t) = \left((\mathbb{I} - \alpha^2 D^2)^{-1} \underline{v}(t)\right)_j, \tag{4.27b}$$

$$j = 1, \ldots, N,$$

where $D_0 h_j$, $D_0 v_j$ and D^2 have been defined in (4.8), (4.11), (4.9) and (4.12). We test the scheme with the initial data (4.13), (4.17), (4.18) and (4.19).

We observe that this method neither succeeds in approximating the correct shock speeds and middle states in Riemann Problems 1 and 3 (Figures 4.18 and 4.20). The performance on Riemann Problem 2 is okay but worse compared to Scheme (4.24), as shown in Figure 4.19. The performance of the scheme on the periodic problem is quite good, as long as no shocks develop in the solution. Moreover, we observe more oscillations than in the approximations computed with (4.24) at the same level. This was to be expected by the results in [3] and also since we have averaged only v in this scheme whereas in (4.24) we also filtered the entry h of the matrix.

4.3 Filtering the eigenvalues of the Jacobian matrix of the flux function

4.3.1 Filtering both eigenvalues

Instead of convolving the components of the Jacobian of the flux function in the non-conservative system (4.2) one could also do the eigendecomposition

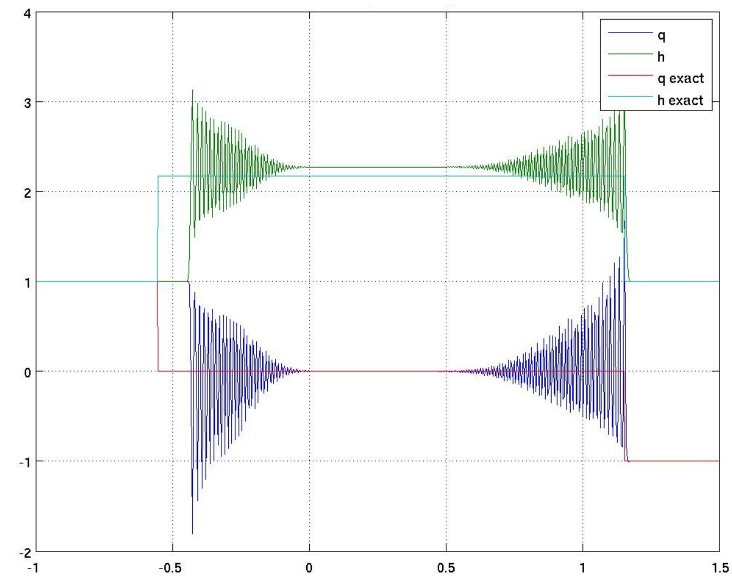

Figure 4.18: Numerical approximation of the initial value problem (4.17) with $\delta = 0.6$ at time $T = 1$ by Scheme (4.27). $\alpha = 0.2$, 800 meshpoints.

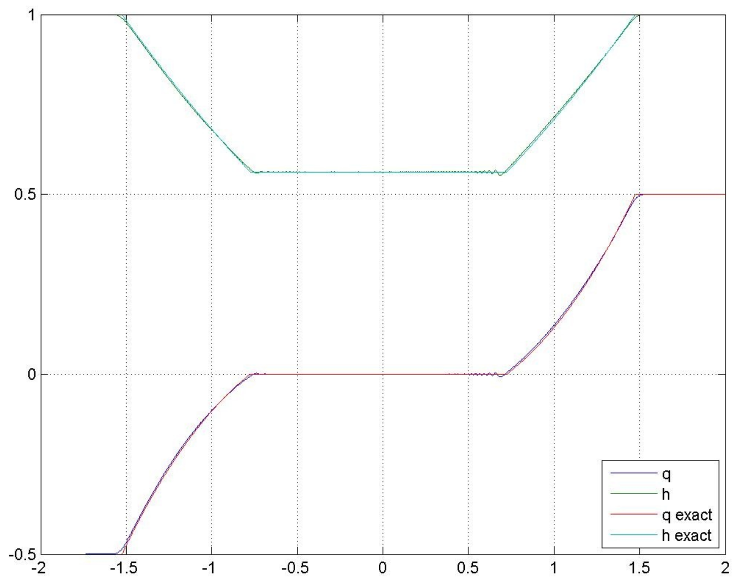

Figure 4.19: Numerical approximation of the initial value problem (4.18) with $\delta = 0.05$ at time $T = 1$ by Scheme (4.27). $\alpha = 0.0125$, 1600 meshpoints, $\mathcal{E}^{\Delta x} = 0.0166$.

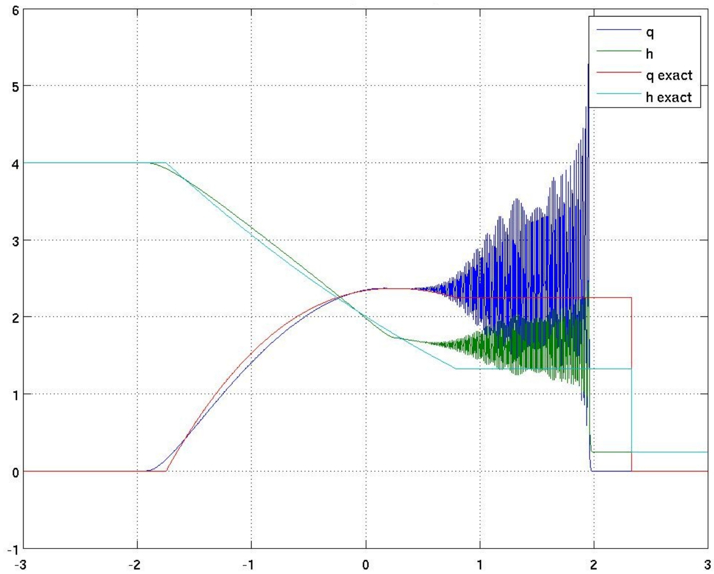

Figure 4.20: Numerical approximation of the initial value problem (4.19) with $\delta = 0.5$ at time $T = 1$ by Scheme (4.27). $\alpha = 0.2$, 1600 meshpoints.

of the system and filter the eigenvalues. The Jacobian of the flux function $f(\mathbf{u}) = (q, q^2/h + h^2/2)^T$ has the eigenvalues

$$\lambda_1(\mathbf{u}) = \frac{q}{h} - \sqrt{h}, \quad \text{and} \quad \lambda_2(\mathbf{u}) = \frac{q}{h} + \sqrt{h}, \tag{4.28}$$

and the eigenvectors,

$$r_1(\mathbf{u}) = \begin{pmatrix} 1 \\ \lambda_1(\mathbf{u}) \end{pmatrix}, \quad \text{and} \quad r_2(\mathbf{u}) = \begin{pmatrix} 1 \\ \lambda_2(\mathbf{u}) \end{pmatrix}. \tag{4.29}$$

Convolving both eigenvalues of the Jacobian with g_α we obtain the equations,

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} 1 & 1 \\ \lambda_1(u) & \lambda_2(u) \end{pmatrix} \begin{pmatrix} \overline{\lambda_1(\mathbf{u})} & 0 \\ 0 & \overline{\lambda_2(\mathbf{u})} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_1(\mathbf{u}) & \lambda_2(\mathbf{u}) \end{pmatrix}^{-1} \begin{pmatrix} h \\ q \end{pmatrix}_x = 0, \tag{4.30}$$

where

$$\overline{\lambda_k(\mathbf{u})}(x) = \big(g_\alpha * \lambda_k(\mathbf{u})\big)(x).$$

If we calculate the matrix multiplications explicitely, we can rewrite (4.30) as

$$
\begin{aligned}
h_t + \overline{v} h_x - v \frac{\overline{\sqrt{h}}}{\sqrt{h}} h_x + \frac{\overline{\sqrt{h}}}{\sqrt{h}} q_x &= 0, \\
q_t + \frac{\overline{\sqrt{h}}}{\sqrt{h}} (h - v^2) h_x + \overline{v} q_x + v \frac{\overline{\sqrt{h}}}{\sqrt{h}} q_x &= 0.
\end{aligned}
$$

Applying the product rule to the terms $q_x = (vh)_x$ and $q_t = (vh)_t$ and using some algebra, we obtain evolution equations for h and v. These coincide with a special case of the equations investigated in [19] which served as a regularization of the homentropic Euler equations. In that paper, global existence and uniqueness of the solution to these equations is proved, speeds of the discontinuities and solutions to Riemann problems are calculated. They find numerical examples where the solution of the modified equations behaves very differently from the one of the homentropic Euler equations. We conduct some numerical experiments to find out whether the same happens if we approximate the equations with a finite difference scheme and a spectral scheme respectively. We denote

$$
R(\mathbf{u}) = \begin{pmatrix} 1 & 1 \\ \lambda_1(\mathbf{u}) & \lambda_2(\mathbf{u}) \end{pmatrix}, \quad \overline{\Lambda}(\mathbf{u}) = \begin{pmatrix} \overline{\lambda_1(\mathbf{u})} & 0 \\ 0 & \overline{\lambda_2(\mathbf{u})} \end{pmatrix} \tag{4.31}
$$

and their discrete counterparts

$$
R_j(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1(\mathbf{u}_j)(t) & \lambda_2(\mathbf{u}_j)(t) \end{pmatrix}, \quad \overline{\Lambda}_j(t) = \begin{pmatrix} \overline{\lambda_1(\mathbf{u}_j)}(t) & 0 \\ 0 & \overline{\lambda_2(\mathbf{u}_j)}(t) \end{pmatrix}, \tag{4.32}
$$

$\mathbf{u}_j(t) := (h_j(t), q_j(t))^T$, $\underline{\lambda}_k(t) := (\dots, \lambda_k(\mathbf{u}_j)(t), \dots)$ and

$$
\overline{\lambda_k(\mathbf{u}_j)}(t) = \left((\mathbb{I} - \alpha^2 D^2)^{-1} \underline{\lambda}_k(t) \right)_j
$$

with D^2 defined in (4.9), (4.12) respectively. The semidiscrete formulation thus reads

$$
\begin{aligned}
\frac{d}{dt} \mathbf{u}_j(t) &= -R_j(t)\, \overline{\Lambda}_j(t)\, R_j(t)^{-1}\, D_0 \mathbf{u}_j(t) := \mathcal{L}(\mathbf{u})_j(t), \\
j &= 1, \dots, N,
\end{aligned} \tag{4.33}
$$

where $D_0 \mathbf{u}_j$ and D^2 have been defined in (4.8), (4.11), (4.9) and (4.12). We test the scheme on initial data (4.13), (4.17), (4.18) and (4.19).

Considering the approximations to the Riemann problems 1 and 3 in Figures 4.21 and 4.23, we can draw the same conclusions as in [19], that is, the shock speeds and the middle states in the Riemann problems are not

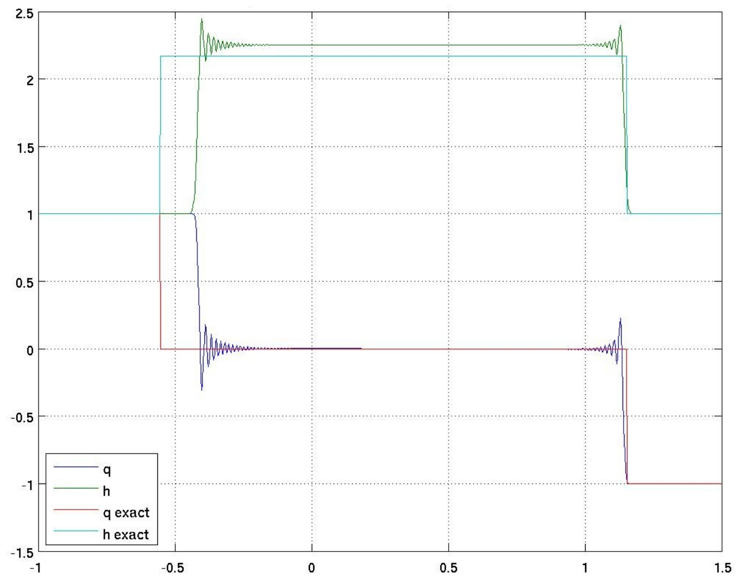

Figure 4.21: Numerical approximation of the initial value problem (4.17) with $\delta = 0.6$ at time $T = 1$ by Scheme (4.33). $\alpha = 0.2$, 800 meshpoints.

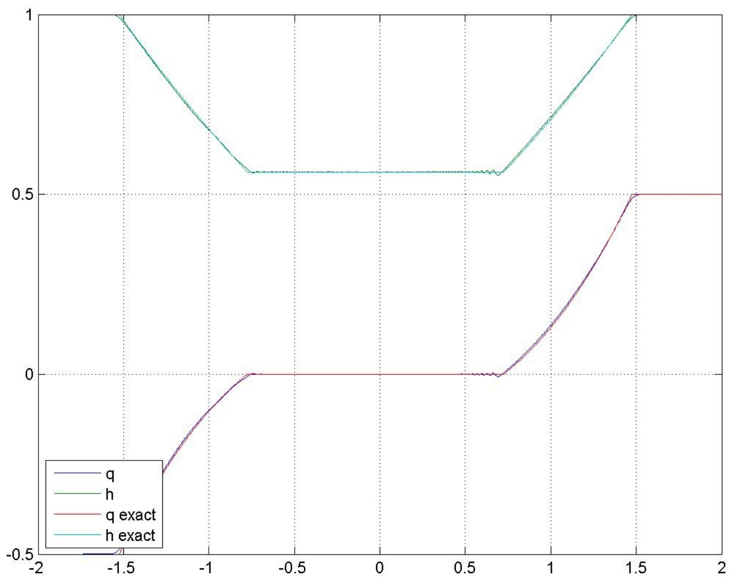

Figure 4.22: Numerical approximation of the initial value problem (4.18) with $\delta = 0.05$ at time $T = 1$ by Scheme (4.33). $\alpha = 0.0125$, 1600 meshpoints, $\mathcal{E}^{\Delta x} = 0.0158$.

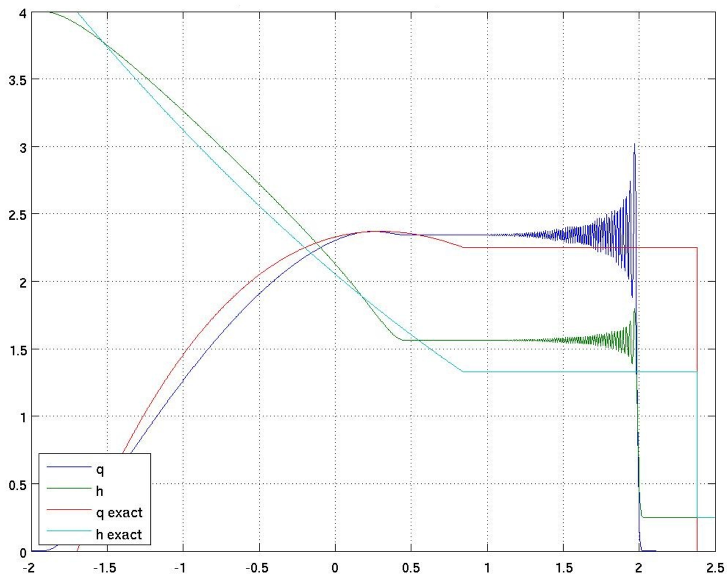

Figure 4.23: Numerical approximation of the initial value problem (4.19) with $\delta = 0.6$ at time $T = 1$ by Scheme (4.33). $\alpha = 0.2$, 1600 meshpoints.

approximated correctly. As for Riemann Problem 2, the scheme gives a satisfactory approximation (Figure 4.22). The same holds for the approximation to the initial value problem with the periodic initial data, again, as long as the solution stays smooth (no picture).

4.3.2 Filtering the velocity v in the eigenvalues

Similar to the previous sections, we can filter only the velocity in the eigenvalues instead of the eigenvalues themselves. This yields the equations,

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} 1 & 1 \\ \lambda_1(\mathbf{u}) & \lambda_2(\mathbf{u}) \end{pmatrix} \begin{pmatrix} \overline{v} - \sqrt{h} & 0 \\ 0 & \overline{v} + \sqrt{h} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_1(\mathbf{u}) & \lambda_2(\mathbf{u}) \end{pmatrix}^{-1} \begin{pmatrix} h \\ q \end{pmatrix}_x = 0, \tag{4.34}$$

with

$$v = \overline{v} - \alpha^2 \, \overline{v}_{xx}.$$

If we compute the product of the matrices explicitely, we see that the above system is equivalent to

$$\begin{aligned} h_t + (\overline{v} - v)h_x + q_x &= 0, \\ q_t + (h - v^2)h_x + (v + \overline{v})q_x &= 0. \end{aligned}$$

Writing this as a system of equations for $\overline{v}$ and h gives

$$h_t + (h\overline{v})_x - \alpha^2 h\overline{v}_{xxx} = 0,$$

$$(\overline{v}h)_t + \left(\frac{h^2}{2}\right)_x + (\overline{v}^2 h)_x - \alpha^2\big((h\overline{v}_{xx})_t + (h\overline{v}\overline{v}_{xx})_x + h\overline{v}\overline{v}_{xxx} - \frac{\alpha^2}{2}(\overline{v}_{xx}^2)_x h\big) = 0.$$

We define the semidiscrete scheme

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} h_j \\ q_j \end{pmatrix}(t) &:= \mathcal{L}(\mathbf{u})_j(t) \\
&= -R_j \begin{pmatrix} (\overline{v}_j - \sqrt{h_j})(t) & 0 \\ 0 & (\overline{v}_j + \sqrt{h_j})(t) \end{pmatrix} R_j^{-1} \begin{pmatrix} D_0 h_j \\ D_0 q_j \end{pmatrix}(t) \\
& j = 1, \dots, N,
\end{aligned} \tag{4.35}$$

where $D_0 h_j$, $D_0 q_j$ and D^2 have been defined in (4.8), (4.11), (4.9) and (4.12). As expected, the approximations to (4.17) and (4.19) turn out to be very oscillatory and do not converge to the exact solution. Therefore we do not include examples here. The approximation to the periodic initial value problem (4.13) and Riemann Problem (4.18) at time $T = 1$ are satisfactory as expected.

Remark 4.1 *The schemes tested in this and the previous sections have turned out to be very unstable and oscillatory when used for the Riemann problems 1 and 3 which have solutions with shocks. Therefore we have also tested them in a modified form with additional diffusion of Roe type, that is, we tested schemes of the form*

$$\begin{aligned}
\frac{d}{dt}\mathbf{u}_j(t) = -\frac{1}{2\Delta x}\Big\{ &\overline{DF}(\mathbf{u}_j)(\mathbf{u}_{j+1} - \mathbf{u}_{j-1}) \\
&- |\hat{\mathbf{A}}_{\mathbf{j+1/2}}|(\mathbf{u}_{j+1} - \mathbf{u}_j) + |\hat{\mathbf{A}}_{\mathbf{j-1/2}}|(\mathbf{u}_j - \mathbf{u}_{j-1})\Big\},
\end{aligned} \tag{4.36}$$

where $\overline{DF}(\mathbf{u}_j)$ denotes the Jacobian matrix of f with entries or eigenvalues filtered and $\hat{\mathbf{A}}_{\mathbf{j\pm 1/2}}$ the Roe matrix. Indeed, the oscillations disappear, but the approximations still differ considerably from the exact solution as it can be seen in the following picture 4.24 for Scheme (4.33). The same wrong behavior of the approximation can be observed if other types of numerical diffusion operators are used in (4.36) in place of the Roe diffusion. In particular, we have tested a Lax-Friedrich type diffusion and a diffusion operator mimicking the physical diffusion of the shallow water equations,

$$\begin{aligned}
\frac{d}{dt}\mathbf{u}_j(t) = -\frac{1}{2\Delta x}\Big\{ &\overline{DF}(\mathbf{u}_j)(\mathbf{u}_{j+1} - \mathbf{u}_{j-1}) \\
&- \begin{pmatrix} 0 \\ \frac{h_{j+1}+h_j}{2}(v_{j+1} - v_j) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{h_j+h_{j-1}}{2}(v_j - v_{j-1}) \end{pmatrix}\Big\},
\end{aligned}$$

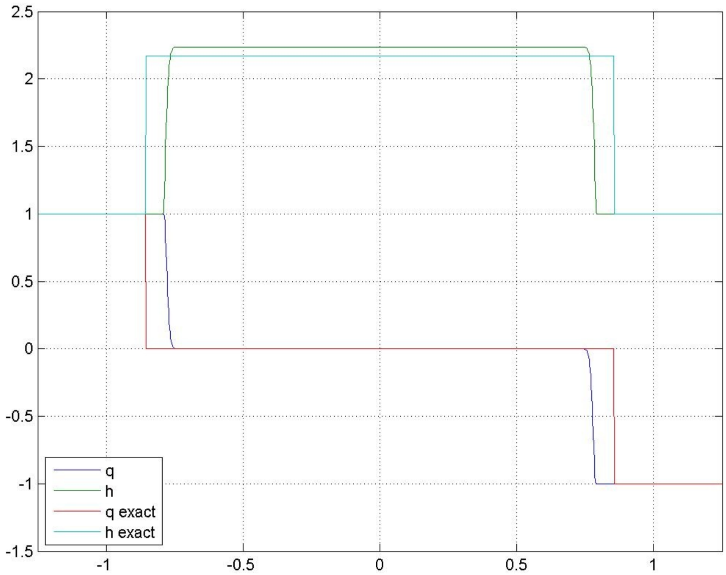

Figure 4.24: Numerical approximation of the initial value problem (4.14) at time $T = 1$ by Scheme (4.33) with Roe diffusion. $\alpha = 0.05$, 1600 meshpoints.

We obtain a satisfactory approximation only with the physical diffusion and when α is so small that the diffusion dominates. We compare this result to the scalar case of the convectively filtered Burgers' equation: In [6], Coclite and Karlsen have shown that the solution to the convectively filtered Burgers' equation with additional diffusion

$$u_t^{\alpha,\varepsilon} + \overline{u}^{\alpha,\varepsilon} u_x^{\alpha,\varepsilon} = \varepsilon\, u_{xx}^{\alpha,\varepsilon},$$
$$\overline{u}^{\alpha,\varepsilon} - \alpha^2 \overline{u}_{xx}^{\alpha,\varepsilon} = u^{\alpha,\varepsilon}$$

converges to the entropy solution of inviscid Burgers' equation if $\alpha = o(\varepsilon)$ and to a weak solution of Burgers' equation for any ratio ε to α. Our experiments indicate that this might be worse for systems of convervation laws: We do not even obtain convergence to a weak solution if α is too large in comparison to ε.

4.4 Conservative methods

One could argue that the reason why the methods tested in the previous sections do not work is that they are not conservative. Therefore we test here the following conservative regularizations which resemble the method suggested in [18] in the context of the one-dimensional Euler equations,

$$h_t + \overline{h}v_x + \overline{v}h_x = 0, \tag{4.37a}$$

$$q_t + \overline{q}v_x + \overline{v}q_x + \overline{h}h_x = 0, \tag{4.37b}$$

and

$$h_t + \overline{h}v_x + \overline{v}h_x = 0, \tag{4.38a}$$

$$q_t + \overline{q}v_x + \overline{v}q_x + \left(\frac{h^2}{2}\right)_x = 0. \tag{4.38b}$$

As before, we denoted $v := q/h$ and $\overline{c} = g_\alpha * c$ for $c \in \{v, q, h\}$. Note that the two models differ only in the second equation, where the term $(h^2/2)_x$ is replaced by $\overline{h}h_x$ in the first model. With a bit of algebra, we can rewrite these equations only in terms of the filtered quantities as

$$\overline{h}_t + (\overline{h}\overline{v})_x = -3\alpha^2 \overline{(\overline{v}_x\overline{h}_x)_x}, \tag{4.39a}$$

$$\overline{q}_t + \left(\overline{q}\overline{v} + \frac{\overline{h}^2}{2}\right)_x = -3\alpha^2 \overline{\left(\overline{q}_x\overline{v}_x + \frac{\overline{h}_x^2}{2}\right)_x}. \tag{4.39b}$$

and

$$\overline{h}_t + (\overline{h}\overline{v})_x = -3\alpha^2 \overline{(\overline{v}_x\overline{h}_x)_x}, \tag{4.40a}$$

$$\overline{q}_t + \left(\overline{q}\overline{v} + \frac{\overline{h^2}}{2}\right)_x = -3\alpha^2 \overline{(\overline{q}_x\overline{v}_x)_x}. \tag{4.40b}$$

For the numerical experiments, we use finite difference schemes for the problems with Neumann boundary conditions and a spectral scheme for the problems with periodic boundary conditions to approximate the solutions of equations (4.37), (4.38), (4.39) and (4.40). We have also tested a pseudo-spectral method as suggested [18] for these equations but we have not obtained better results.

Approximation of the models (4.37) and (4.38)

In a first step, we analyze and compare the schemes (4.41) and (4.42). System (4.37) is approximated by the scheme

$$\frac{d}{dt}\begin{pmatrix} h_j \\ q_j \end{pmatrix}(t) = -\begin{pmatrix} \overline{v}_j D_0 h_j + \overline{h}_j D_0 v_j \\ \overline{q}_j D_0 v_j + \overline{v}_j D_0 q_j + \overline{h}_j D_0 h_j \end{pmatrix}(t) := \mathcal{L}(\mathbf{u})_j(t), \tag{4.41a}$$

$$\overline{c}_j(t) = \left((\mathbb{I} - \alpha^2 D^2)^{-1} \underline{c}(t)\right)_j, \quad c \in \{h, v, q\} \tag{4.41b}$$

$$j = 1, \dots, N,$$

where $D_0 h_j$, $D_0 v_j$, $D_0 q_j$ and D^2 have been defined in (4.8), (4.11), (4.9) and (4.12). Similarly, for (4.38), we use the scheme

$$\frac{d}{dt}\begin{pmatrix} h_j \\ q_j \end{pmatrix}(t) = -\begin{pmatrix} \overline{v}_j D_0 h_j + \overline{h}_j D_0 v_j \\ \overline{q}_j D_0 v_j + \overline{v}_j D_0 q_j + \frac{1}{2} D_0(h_j^2) \end{pmatrix}(t) := \mathcal{L}(\mathbf{u})_j(t), \tag{4.42a}$$

$$\overline{c}_j(t) = \left((\mathbb{I} - \alpha^2 D^2)^{-1} \underline{c}(t)\right)_j, \quad c \in \{h, v, q\} \tag{4.42b}$$

$$j = 1, \ldots, N.$$

We test the schemes with initial data (4.13), (4.17), (4.18) and (4.19). We

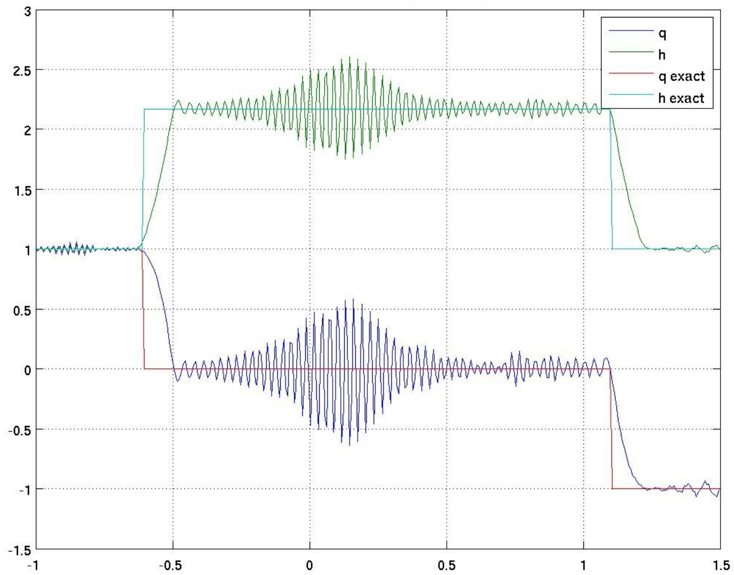

Figure 4.25: Numerical approximation of the initial value problem (4.17) by (4.41) with $\delta = 0.5$ at time $T = 1$. $\alpha = 0.1$, 400 meshpoints.

see in Figures 4.25 and 4.26 that both approximations yield the correct middle state of the height. The gradients in the regions connecting the left and middle and the middle and right state steepen as α decreases, but the approximations become so oscillatory that they finally blow up. The same happens if we increase the number of meshpoints. Furthermore, these two schemes turn out to be very unstable if used to approximate the Riemann problems (4.18) and (4.19). The approximations to (4.19) do not behave like the exact solution at all (we have not included the pictures). However, the approximations to the periodic initial value problem (4.13) are satisfactory as long as the exact solution does not develop shocks (no Figure).

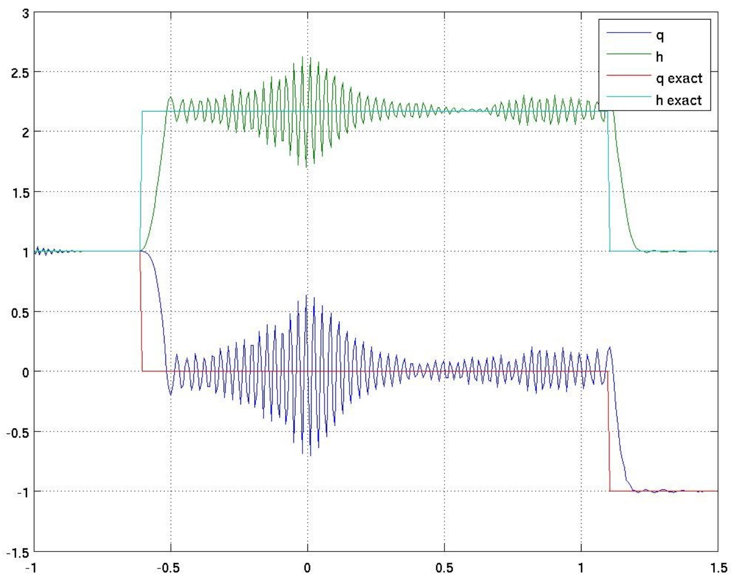

Figure 4.26: Numerical approximation of the initial value problem (4.17) by (4.42) with $\delta = 0.5$ at time $T = 1$. $\alpha = 0.1$, 400 meshpoints.

Approximation of (4.39) and (4.40)

In a second step, we test Scheme (4.46) combined with the fluxes (4.45) and (4.47) and their spectral counterparts. To approximate (4.39), we simplify the notation by denoting by $h_j(t)$ the approximation of $\overline{h}(x_j,t)$, by $q_j(t)$ the approximation to $\overline{q}(x_j,t)$ and by $v_j(t)$ the approximation to $\overline{v}(x_j,t)$. Notice that

$$\overline{v} = g_\alpha * \left(\frac{\overline{q} - \alpha^2 \overline{q}_{xx}}{\overline{h} - \alpha^2 \overline{h}_{xx}}\right).$$

In order to find the same expression for the discrete quantities $v_j(t)$, we simplify the notation for this section by defining the discrete Helmholtz operator $(\mathbb{I} - \alpha^2 D^2)$ by

$$\mathcal{A}\,\underline{c} := (\mathbb{I} - \alpha^2 D^2)\,\underline{c}, \quad \mathcal{A}\,c_j := \big((\mathbb{I} - \alpha^2 D^2)\,\underline{c}\big)_j, \quad c \in \{q, h, v, \dots\} \tag{4.43}$$

with D^2 given in (4.9) and (4.12), and similarly its inverse

$$\mathcal{A}^{-1}\underline{c} := (\mathbb{I} - \alpha^2 D^2)^{-1}\underline{c}, \quad \mathcal{A}^{-1}c_j := \big((\mathbb{I} - \alpha^2 D^2)^{-1}\underline{c}\big)_j, \quad c \in \{q, h, v, \dots\}. \tag{4.44}$$

Then v_j is given by

$$v_j(t) = \mathcal{A}^{-1}\left(\frac{\mathcal{A}\, q_j}{\mathcal{A}\, h_j}\right)_j(t).$$

We omit the dependencies of the variables on time and denote $f(\mathbf{u}_j) := (h_j v_j, q_j v_j + h_j^2/2)^T$. Moreover, we let

$$D_+ c_j := \frac{1}{\Delta x}(c_{j+1} - c_j), \quad D_- c_j := \frac{1}{\Delta x}(c_j - c_{j-1}), \quad c \in \{v, q, h, \dots\},$$

denote the forward and backward differences respectively. We use the same notation for vectors of approximated quantities:

$$D_+ \underline{c} := \frac{1}{\Delta x}(c_2 - c_1, \dots, c_{j+1} - c_j, \dots c_{N+1} - c_N)^T, \quad c \in \{v, q, h, \dots\},$$

$$D_+ \underline{c}\, D_+ \underline{d} := \frac{1}{\Delta x^2}(((c_2 - c_1) \cdot (d_2 - d_1), \dots, (c_{N+1} - c_N) \cdot (d_{N+1} - d_N))^T, \qquad c, d \in \{v, q, h, \dots\},$$

(and similarly for the backward differences D_-). Now we can define the numerical flux

$$\mathbf{F}_{j+1/2} = \frac{f(\mathbf{u}_{j+1}) + f(\mathbf{u}_j)}{2} + 3\alpha^2 \begin{pmatrix} \mathcal{A}^{-1}(D_+ \underline{v}\, D_+ \underline{h})_j \\ \mathcal{A}^{-1}(D_+ \underline{q}\, D_+ \underline{v} + \frac{1}{2} D_+(\underline{h^2}))_j \end{pmatrix}. \tag{4.45}$$

We define the semidiscrete numerical scheme

$$\frac{d}{dt}\mathbf{u}_j(t) = -\frac{1}{\Delta x}(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2})(t), \tag{4.46}$$

which we will use to approximate (4.39) in the case of Neumann boundary conditions. For (4.40) we will use (4.46) with $\mathbf{F}_{j+1/2}$ defined by

$$\mathbf{F}_{j+1/2} = \frac{\widetilde{f}(\mathbf{u}_{j+1}) + \widetilde{f}(\mathbf{u}_j)}{2} + 3\alpha^2 \begin{pmatrix} \mathcal{A}^{-1}(D_+ \underline{v}\, D_+ \underline{h})_j \\ \mathcal{A}^{-1}(D_+ \underline{q}\, D_+ \underline{v})_j \end{pmatrix}, \tag{4.47}$$

where $\widetilde{f}(\mathbf{u_j})$ is given by

$$\widetilde{f}(\mathbf{u}_j) := \begin{pmatrix} h_j v_j \\ q_j v_j + \frac{1}{2}\widetilde{h_j^2} \end{pmatrix},$$
$$\widetilde{h_j^2} := \mathcal{A}^{-1}\big((\mathcal{A}\, h_j)^2\big)_j \tag{4.48}$$

For the initial value problems with periodic boundary conditions, we use a spectral method for the approximation of the derivatives, that is for (4.39), we employ the scheme

$$\frac{d}{dt}h_j(t) = -D_0(v_j h_j) - 3\alpha^2 D_0\big(\mathcal{A}^{-1}(D_0 v_j \cdot D_0 h_j)_j\big), \tag{4.49a}$$

$$\frac{d}{dt}q_j(t) = -D_0\left(q_j v_j + \frac{h_j^2}{2}\right) - 3\alpha^2 D_0\Big(\mathcal{A}^{-1}\big(D_0 q_j \cdot D_0 v_j + \frac{1}{2}(D_0 h_j)^2\big)_j\Big), \tag{4.49b}$$

$$j = 1, \ldots, N$$

where D_0 denotes a spectral approximation of the first derivative, given in (4.10).

For (4.40) and periodic boundary conditions, we use the scheme

$$\frac{d}{dt}h_j(t) = -D_0(v_j h_j) - 3\alpha^2 D_0\big(\mathcal{A}^{-1}(D_0 v_j \cdot D_0 h_j)_j\big), \tag{4.50a}$$

$$\frac{d}{dt}q_j(t) = -D_0\left(q_j v_j + \frac{\widetilde{h_j^2}}{2}\right) - 3\alpha^2 D_0\Big(\mathcal{A}^{-1}\big(D_0 q_j \cdot D_0 v_j\big)_j\Big), \tag{4.50b}$$

$$j = 1, \ldots, N$$

with $\widetilde{h_j^2}$ defined in (4.48). We test the schemes with initial data (4.13), (4.17), (4.18) and (4.19).

As it can be seen Figure 4.27 the approximation with (4.46) and (4.47) of the middle state and the shock speeds appear to be correct for initial data (4.14). However, for increasing N or decreasing α the scheme tends to become unstable. Scheme (4.46) with (4.45) shows a very similar behavior. As far as initial value problem (4.18) is concerned, the performance of the schemes is very bad. To prevent a blow up, we have to choose the smoothing parameter of the intial data δ large compared to the schemes tested in the previous sections. This decreases the accuracy additionally. In contrast to what one might have expected, the scheme with flux (4.45), where the height h is regularized in the second equation, is even more unstable than the scheme with flux (4.47). In Figure 4.28, an approximation of the solution at time $T = 0.5$ by Scheme (4.46) with (4.47) can be seen. The approximations to the dam break problem (4.19) are not satisfactory either: In Figure 4.29 we see that the approximation with (4.46) and (4.47) at time $T = 0.5$ with $\alpha = 0.05$ and $N = 800$ behaves different from the exact solution and for larger N and smaller α the approximation blows up. In [18] a possible explanation for the unstable behavior of the scheme is given; they explain that it could be caused by the term $\overline{v}_x\overline{h}_{xx}$ which we obtain when applying the product rule to the term $-3\alpha^2\overline{(\overline{v}_x\overline{h}_x)_x}$ on the right hand side of equations (4.39a) and

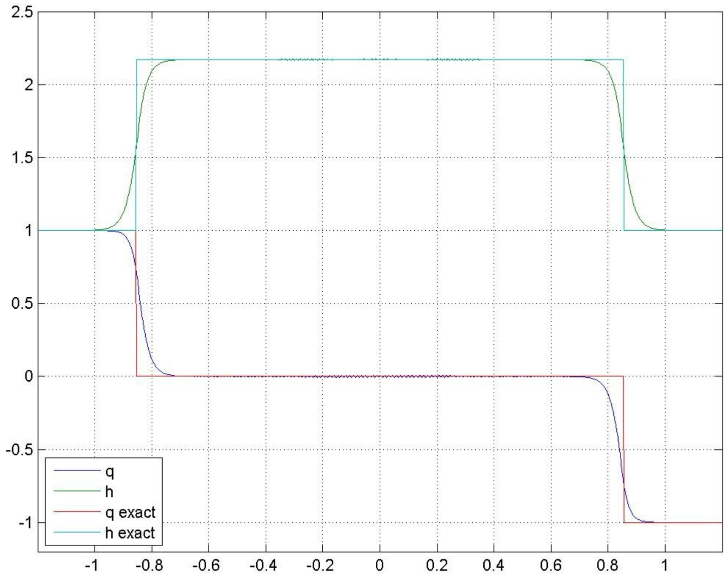

Figure 4.27: Numerical approximation of the initial value problem (4.14) by (4.46) and (4.47) at time $T = 1$. $\alpha = 0.025$, 800 meshpoints.

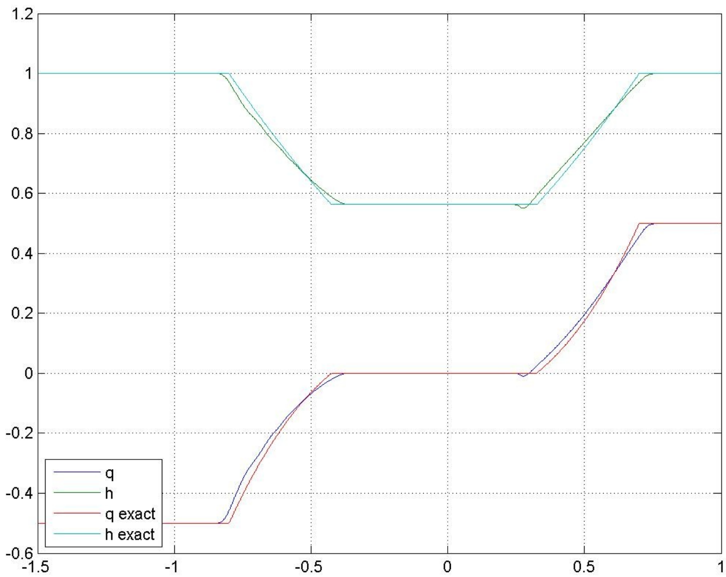

Figure 4.28: Numerical approximation of the initial value problem (4.18) by (4.46) and (4.47) with $\delta = 0.1$ at time $T = 0.5$. $\alpha = 0.01$, 1600 meshpoints.

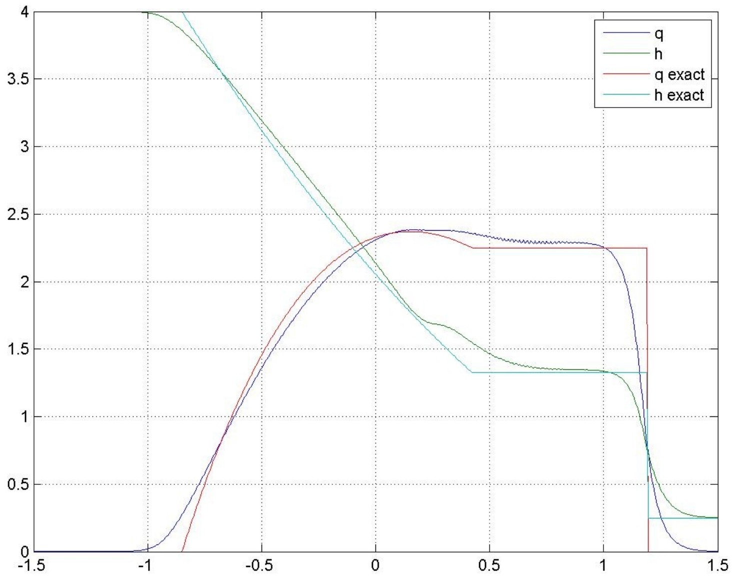

Figure 4.29: Numerical approximation of the initial value problem (4.19) by (4.46) and (4.47) with $\delta = 0.3$ at time $T = 0.5$. $\alpha = 0.01$, 800 meshpoints.

(4.40a). If $\overline{v}_x > 0$, this term resembles a negative viscosity which is known to have a destabilizing effect. This might cause the instabilities in the approximations of (4.18) and (4.19). Similar to the spectral schemes investigated previously, the spectral schemes for the conservative regularizations (4.39) and (4.40) give an accurate approximation to the periodic problem as long as the solution stays smooth. Nevertheless, they turn out to be very unstable when the exact solution develops discontinuities.

Remark 4.2 *We have also conducted experiments with Schemes (4.7) and (4.41) and the Helmholtz filter replaced by the compact filter*

$$g_\alpha(x) = \frac{1}{2\alpha}\chi_{[-\alpha,\alpha]}(x).$$

These modified schemes appear to be even more unstable. The approximations by (4.7) turn out to be much more oscillatory than those of the same scheme with the Helmholtz filter and the middle state of the height in Riemann Problem (4.17) is not correct either. The approximations with the modified scheme (4.41) are very oscillatory and tend to blow up for many choices of α and N. Nevertheless, the middle state of the height in the approximation to Riemann Problem (4.17) seems to be at the correct level.

Remark 4.3 *Another conservative regularization of the shallow water equations which we have tested is*

$$h_t + q_x = 0, \tag{4.51a}$$
$$q_t + \overline{q} v_x + \overline{v} q_x + \overline{h} h_x = 0, \tag{4.51b}$$

which can be written in terms of the filtered quantities as

$$\overline{h}_t + \overline{q}_x = 0, \tag{4.52a}$$
$$\overline{q}_t + \left(\overline{q}\overline{v} + \frac{\overline{h}^2}{2} \right)_x = -3\alpha^2 \overline{\left(\overline{q}_x \overline{v}_x + \frac{\overline{h}_x^2}{2} \right)_x}. \tag{4.52b}$$

In a similar way, we used finite differences and spectral methods to approximate the above equations. In contrast to the approximations to (4.37), (4.38), (4.39) and (4.40), we obtain a wrong middle state for the height in initial value problem (4.17). Hence conservation of the mass and the momentum by the regularized equation is not enough to guarantee convergence to the exact solution.

4.5 The effect of filtering variables in finite volume schemes

The numerical results of the previous sections of this chapter have not convinced us that filtering variables in the non-conservative forms of the shallow water equations (4.1) is a useful technique to regularize the equations. For this reason we investigate a different approach in this section, that is we filter the velocity v in the conservative form of the shallow water equations (4.1),

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} \overline{v} h \\ \overline{v}^2 h + \frac{h^2}{2} \end{pmatrix}_x = 0 \tag{4.53a}$$
$$\begin{pmatrix} h \\ q \end{pmatrix}(x, 0) = \begin{pmatrix} h_0 \\ q_0 \end{pmatrix}(x), \tag{4.53b}$$
$$\overline{v}(x) = g_\alpha * v(x), \tag{4.53c}$$

where g_α is either the Helmholtz filter (4.4) or the normalized characteristic function of the interval $[-\alpha, \alpha]$,

$$g_\alpha(x) = \frac{1}{2\alpha} \chi_{[-\alpha,\alpha]}(x),$$

which we will call *box filter* in the following. We have conducted numerical experiments for these equations using a standard finite volume scheme with

Roe flux, and v filtered in the flux function. This means that we are adding numerical diffusion at meshsize Δx, i.e. we are approximating (4.53) as the limit $\mu \to 0$ of

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} \overline{v}h \\ \overline{v}^2 h + \frac{h^2}{2} \end{pmatrix}_x = \mu\big(\mathbf{B}(\mathbf{u})\mathbf{u}_x\big)_x, \tag{4.54}$$

where $\big(\mathbf{B}(\mathbf{u})\mathbf{u}_x\big)_x$ is a viscosity term and μ is of size Δx in the numerical scheme. Specifically, we apply the scheme in flux form (4.46) with $\mathbf{F}_{j+1/2}$ defined as

$$\mathbf{F}_{j+1/2} = \frac{\overline{f}(\mathbf{u}_{j+1}) + \overline{f}(\mathbf{u}_j)}{2} - \frac{1}{2}|\hat{\mathbf{A}}_{j+1/2}|(\mathbf{u}_{j+1} - \mathbf{u}_j) \tag{4.55}$$

where $\hat{\mathbf{A}}_{j+1/2}$ is the Roe matrix and

$$\overline{f}(\mathbf{u}_j)(t) := \begin{pmatrix} \overline{v}_j h_j \\ \overline{v}_j^2 h_j + h_j^2/2 \end{pmatrix}(t),$$
$$\overline{v}_j := \big((\mathbb{I} - \alpha^2 D^2)^{-1} \underline{v}(t)\big)_j.$$

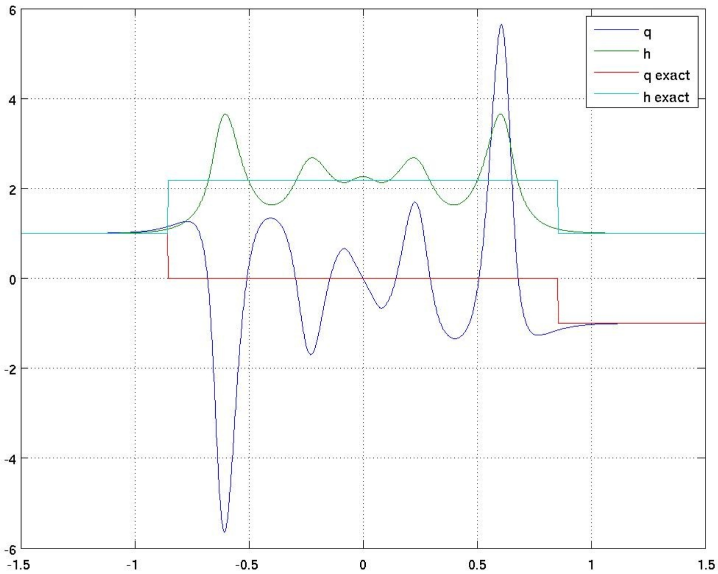

Figure 4.30: Numerical approximation of the initial value problem (4.14) by Scheme (4.46) with (4.55) and the Helmholtzfilter at time $T = 1$. $\alpha = 0.1$, 1600 meshpoints.

The numerical experiments show that, if we choose α large in comparison to the meshsize Δx, the approximations have a very different behavior than

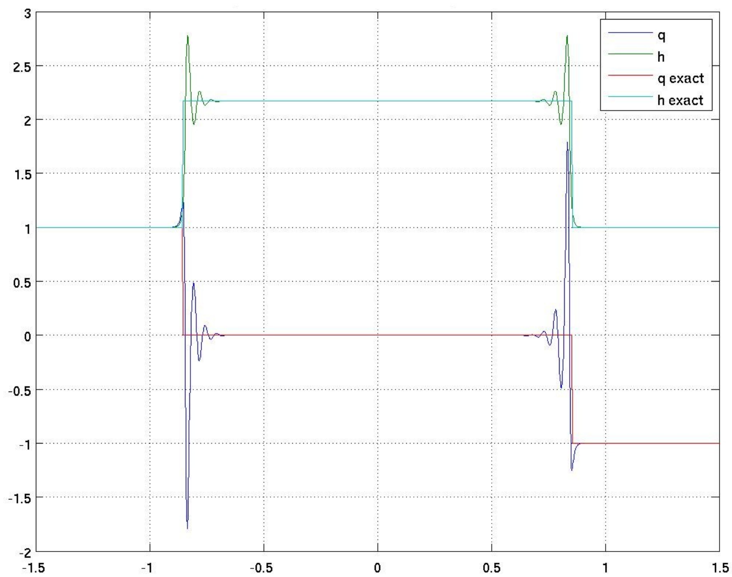

Figure 4.31: Numerical approximation of the initial value problem (4.14) by Scheme (4.46) with (4.55) and the Helmholtzfilter at time $T = 1$. $\alpha = 0.01$, 1600 meshpoints.

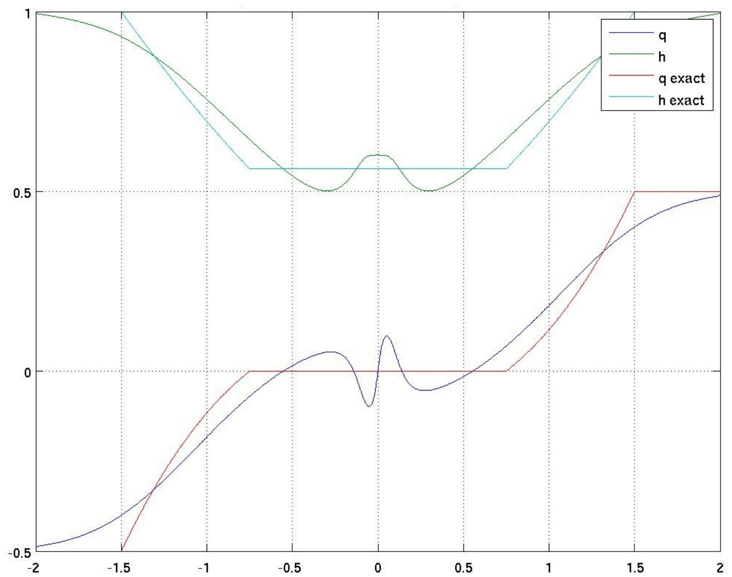

Figure 4.32: Numerical approximation of the initial value problem (4.15) by Scheme (4.46) with (4.55) and the Helmholtzfilter at time $T = 1$. $\alpha = 0.1$, 1600 meshpoints.

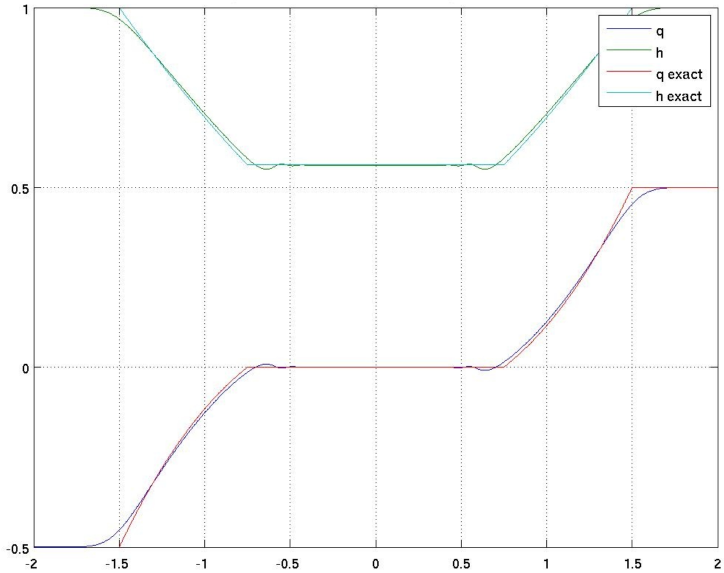

Figure 4.33: Numerical approximation of the initial value problem (4.15) by Scheme (4.46) with (4.55) and the Helmholtzfilter at time $T = 1$. $\alpha = 0.01$, 1600 meshpoints.

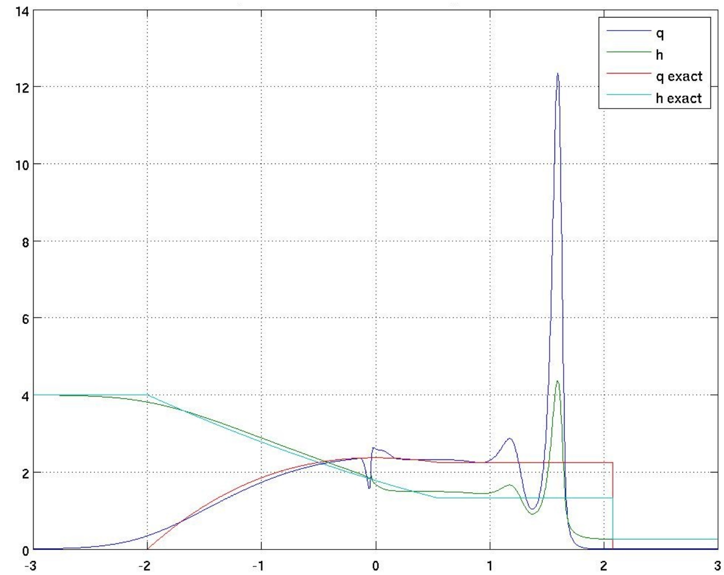

Figure 4.34: Numerical approximation of the initial value problem (4.16) by Scheme (4.46) with (4.55) and the Helmholtzfilter at time $T = 1$. $\alpha = 0.1$, 1600 meshpoints.

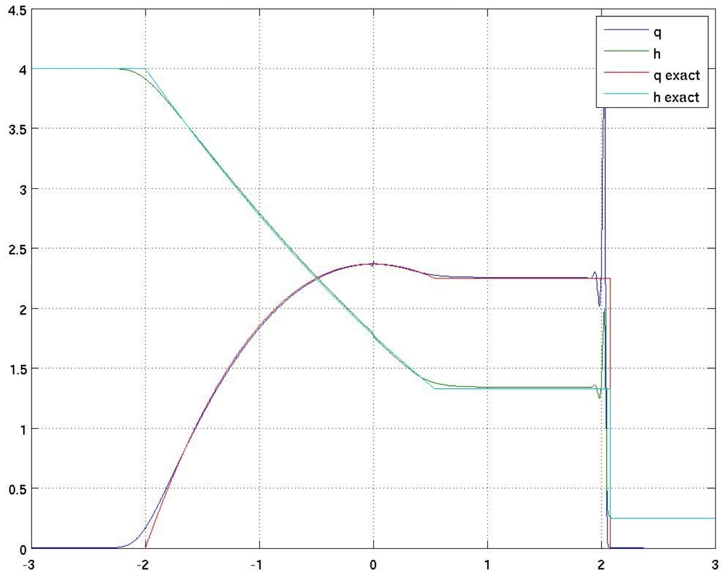

Figure 4.35: Numerical approximation of the initial value problem (4.16) by Scheme (4.46) with (4.55) and the Helmholtzfilter at time $T = 1$. $\alpha = 0.01$, 1600 meshpoints.

the exact solution. They show large oscillations which become smaller in amplitude and restrict to a smaller area as α decreases. However, the approximation seems to converge to the exact solution as α decreases (except in the case of the dam break problem (4.16), where the standard Roe scheme without entropy fix gives a wrong solution). This is probably because for small α the numerical diffusion of the Roe scheme dominates. The scheme with the box filter gives only stable approximations for small α. Interestingly, for very small α (corresponding to a support of three cells of the box filter), we obtain a better approximation of the correct solution of the dam break problem (4.16) in the region arount $x = 0$ than the Roe scheme gives, which can be observed in Figure 4.36. Apart from this last example, it seems that filtering the velocity in the conservative form of the shallow water equations (4.1) results in a very different behavior compared to the one we expect from the solution of the shallow water equations.

4.6 Conclusions

We have conducted experiments with numerical schemes for several modifications of the shallow water equations with nonlocal quantities. In most cases the averaging of quantities fails to stabilize the numerical schemes if

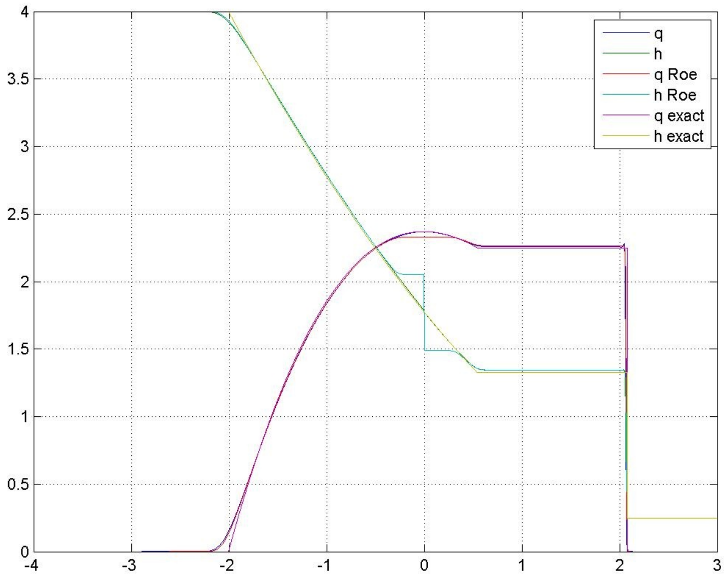

Figure 4.36: Numerical approximation of the initial value problem (4.16) by Scheme (4.46) with (4.55) and the box filter at time $T = 1$. $\alpha = 0.005625$, 1600 meshpoints.

shocks develop in the solution and more importantly, for most schemes the approximations do not converge to the exact solution of the equations. We have only obtained meaningful approximations of Riemann Problem 1 with the numerical schemes for (4.37) and (4.38). These equations satisfy conservation of the mass and the momentum but still, this cannot be the only reason for the better approximation as numerical experiments conducted for other conservative modifications of the shallow water equations have shown. Moreover, these schemes fail to capture the correct behavior of the solution in the case of the dam break problem (4.16) too. A reason for the incorrect behavior of most of the approximations could be that the different 'regularizations' of the shallow water equations are based on the non-conservative forms (4.2) and (4.3) of the equations. It has been observed previously that this constitutes also a problem for viscous regularizations of non-conservative systems insofar as different numerical diffusion operators yield different behavior of the approximated solutions [11]. Furthermore, we have already observed in Chapter 2 in the case of the scalar equation

$$u_t^\alpha + \overline{f'(u^\alpha)}u_x^\alpha = 0,$$

for a general flux f, that the limit function u obtained when letting $\alpha \to 0$ is not automatically a weak solution of inviscid Burgers' equation since the shock speed can be wrong. The same issue might occur here.

From the experiments in Section 4.5 we conclude that even filtering in the conservative form of the equations can result in totally different behavior of the solution.

Chapter 5

Conclusions

We have analyzed the convectively filtered Burgers' equation in Chapter 2 and found that the limit u of its solutions $(u^\alpha)_{\alpha>0}$ cannot satify an entropy inequality owing to the time reversibility of the equation. In spite of this, it has been conjecuted that the sequence $(u^\alpha)_{\alpha>0}$ converges to the entropy solution if the initial data is continuous. We have not made advances in terms of proving this conjecture.

In Section 2.2 we show the non-existence of an L^1-contraction estimate which is uniform with respect to α, by presenting a counterexample. This implies that it is very hard to find a numerical scheme which converges independently of the parameter α to the exact solution. An open question remains whether there exists a contraction estimate with respect to the L^∞-norm of the initial data, such as (2.26).

In Chapter 3 we investigate the CFB equation with a particular filter depending on time trying to overcome the reversibility of the equation which made convergence to the entropy solution impossible. We have found that the sequence of solutions $(u^\alpha)_{\alpha>0}$ of this equation converges to a weak solution of Burgers' equation but that this limit does not necessarily satisfy the entropy inequality either.

Having seen that the solution of the CFB equation converges at least to a weak solution of Burgers' equation, we wanted to know whether similar facts hold for systems of conservation laws with filtered solution quantities. We have investigated this question numerically for the particular case of the shallow water equations. We have found that for most of the tested modifications of the system with filtered solution quantities, this does not seem to be the case. Furthermore, some of the tested models even fail to regularize the solution of the conservation law in the sense that they would smoothen discontinuities present in the solution of the unfiltered system.

That finding the 'right' regularization with filtered variables can be a delicate issue, has already been indicated in Section 2.4 where we have found that the solution to Equation (2.29a) with filtered derivative of the flux function does not converge to a weak solution of the corresponding conservation law when letting $\alpha \to 0$.

In summary, the results of this thesis indicate that filtering of variables might not be an appropriate way of regularizing the solutions to systems of conservation laws.

Open questions

As we have already mentioned, the question whether the solutions u^α of the convectively filtered Burgers' equation converge to the entropy solution of Burgers' equation if the initial data is continuous has remained unanswered. Connected to this is the question whether there exists a stability estimate of the form (2.26). This could be helpful for finding a numerical method for the convectively filtered Burgers' equation which converges independently of α.

As far as the numerical methods for the shallow water equations are considered, it would be interesting to further investigate the reasons why certain numerical methods yield good approximations and others do not. This could provide more insight into the effect of filtering variables in systems of conservation laws. Moreover, we have not tested numerical methods for the shallow water system which base on the Riemann invariants of the equations yet. They might prove to be more suitable for approximating the solution of the system than the finite difference and spectral methods employed in Chapter 4.

Appendix A

First order scheme for the convectivley filtered Burgers' equation

In this section, we propose a first order numerical scheme to approximate the convectively filtered Burgers' equation and show its convergence. We restate the convectively filtered Burgers' equation:

$$u_t^\alpha + \overline{u}^\alpha u_x^\alpha = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{A.1a}$$

$$\overline{u}^\alpha = u^\alpha * g_\alpha, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{A.1b}$$

$$u^\alpha(x,0) = u^0(x), \quad x \in \mathbb{R}, \tag{A.1c}$$

where

$$g_\alpha(x) = \frac{1}{\alpha} g\left(\frac{x}{\alpha}\right)$$

is a nonnegative, even, decreasing (with respect to the absolute value of the argument) function with $\int g = 1$, and the initial data $u^0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$.

We discretize the spatial domain by an equidistant grid with gridpoints denoted by $x_{j+\frac{1}{2}}$, $j \in \mathbb{Z}$, cell midpoints denoted by $x_j = j\Delta x$, $j \in \mathbb{Z}$, where Δx is the size of a cell. For Δt chosen such that it satisfies a CFL-condition which we will specify later, we denote by $t^n = n\,\Delta t$, $n = 0, \dots, N$; $N\,\Delta t = T$, the discretization in time. We denote by u_j^n an approximation of $u^\alpha(x_j, t^n)$ and by $\overline{u}_j^n$ an approximation of $\overline{u}^\alpha(x_j, t^n)$. In particular, we approximate $\overline{u}^\alpha(x_j, t^n)$ by the midpoint rule:

$$\overline{u}_j^n = \frac{\Delta x}{\alpha} \sum_{l \in \mathbb{N}} g\left(\frac{x_j - x_l}{\alpha}\right) u_l^n. \tag{A.2}$$

As a CFL-condition, we choose at time $t = t^n$

$$\sup_j |\overline{u}_j^n| \frac{\Delta t}{\Delta x} \leq \frac{1}{3} \tag{A.3}$$

Then we consider the following finite difference scheme:

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{\Delta t}{2\,\Delta x}\overline{u}_j^n(u_{j+1}^n - u_{j-1}^n). \tag{A.4}$$

From the approximations u_j^n, we define the piecewise constant function

$$u^{\Delta x}(x,t) = u_j^n, \quad (x,t) \in [x_{j-1/2}, x_{j+1/2}) \times [t^n, t^{n+1}). \tag{A.5}$$

Proposition A.1 *Under the CFL-condition (A.3), Scheme (A.4) is conservative and for $u^0 \in L^\infty(\mathbb{R})$ maximum preserving independently of α. If $u^0 \in BV(\mathbb{R})$, it is total variation bounded with a bound depending on α, i.e.*

$$TV(u^n) \leq C_\alpha\, TV(u^0),$$

Furthermore, if we choose $u^0 \in C_b^2(\mathbb{R})$ and $g \in C_b^2(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$, $u^{\Delta x}(\cdot, T)$, defined in (A.5), converges uniformly to $u^\alpha(\cdot, T)$, as $\Delta x \to 0$:

$$|u^{\Delta x}(x,T) - u^\alpha(x,T)| \leq C\Delta x, \quad x \in \mathbb{R},$$

where C is a constant depending on α, T, g and $\|u^0\|_{C^2(\mathbb{R})}$, and $\|\cdot\|_{C^2(\mathbb{R})}$ is the norm in $C^2(\mathbb{R})$, that is $\|f\|_{C^2(\mathbb{R})} := \max\{\sup_{x\in\mathbb{R}}|f(x)|, \sup_{x\in\mathbb{R}}|f'(x)|, \sup_{x\in\mathbb{R}}|f''(x)|\}$. Moreover, in this case we have that the first and the second differences $(u_j^n - u_{j-1}^n)/\Delta x$, $(u_{j+1}^n - 2u_j^n + u_{j-1}^n)/\Delta x^2$ respectively, are bounded independently of Δx with a bound depending on the initial data,

$$\frac{|u_j^n - u_{j-1}^n|}{\Delta x} \leq C_{\alpha,u^0,T}\,\|u^0\|_{C^1}, \quad \frac{|u_{j+1}^n - 2u_j^n + u_{j-1}^n|}{\Delta x^2} \leq \widetilde{C}_{\alpha,u^0,T,g}\|u^0\|_{C^2}$$

$j \in \mathbb{Z}$, $n = 1, \ldots, N$ and $\|\cdot\|_{C^1}$ is the norm on $C^1(\mathbb{R})$.

Proof Conservation of $\Delta x \sum_j u_j^n$

We show that the discrete quantity $\Delta x \sum_j u_j^n$ which approximates the integral $\int u^\alpha(x,t^n)dx$ is conserved at every time step. Inserting the right-hand side of (A.4) for u_j^{n+1}, we have

$$
\begin{aligned}
\sum_j u_j^{n+1} &= \frac{1}{2}\sum_j (u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta x}\sum_j \overline{u}_j^n (u_{j+1}^n - u_{j-1}^n) \\
&= \sum_j u_j^n - \frac{\Delta t}{2\Delta x}\sum_j \Delta x \sum_l g_\alpha(x_j - x_l)\, u_l^n (u_{j+1}^n - u_{j-1}^n) \\
&= \sum_j u_j^n - \frac{\Delta t}{2}\sum_j \sum_l g_\alpha(\Delta x(j-l))\, u_l^n (u_{j+1}^n - u_{j-1}^n) \\
&= \sum_j u_j^n - \frac{\Delta t}{2}\sum_j \sum_l (g_\alpha(\Delta x(j-1-l)) - g_\alpha(\Delta x(j+1-l))) u_l^n\, u_j^n \\
&= \sum_j u_j^n - \frac{\Delta t}{2}\left(\sum_{j,l} g_\alpha(\Delta x(j-1-l)) u_l^n\, u_j^n - \sum_{j,l} g_\alpha(\Delta x(j+1-l)) u_l^n\, u_j^n\right) \\
&= \sum_j u_j^n - \frac{\Delta t}{2}\left(\sum_{j,l} g_\alpha(\Delta x(-j+1+l)) u_l^n\, u_j^n - \sum_{j,l} g_\alpha(\Delta x(j+1-l)) u_l^n\, u_j^n\right) \\
&= \sum_j u_j^n - \frac{\Delta t}{2}\left(\sum_{j,l} g_\alpha(\Delta x(j+1-l)) u_j^n\, u_l^n - \sum_{j,l} g_\alpha(\Delta x(j+1-l)) u_l^n\, u_j^n\right) \\
&= \sum_j u_j^n
\end{aligned}
$$

where we used the symmetry of g in the sixth equation and exchanged the summation indices j and l in the second sum of the seventh equation.

Maximum principle:

In order to show that the maximum is preserved, we rewrite the scheme in the following incremental form:

$$u_j^{n+1} = u_j^n + C_{j+\frac{1}{2}}^n (u_{j+1}^n - u_j^n) - D_{j-\frac{1}{2}}^n (u_j^n - u_{j-1}^n), \tag{A.6}$$

where

$$C_{j+\frac{1}{2}}^n = \frac{1}{2} - \frac{\Delta t}{2\Delta x}\overline{u}_j^n, \quad D_{j+\frac{1}{2}}^n = \frac{1}{2} + \frac{\Delta t}{2\Delta x}\overline{u}_{j+1}^n, \tag{A.7}$$

$j \in \mathbb{N}$ and $n = 0, \dots, N$. We observe that, thanks to (A.3),

$$C_{j+\frac{1}{2}}^n, D_{j+\frac{1}{2}}^n \geq 0 \quad \text{and} \quad C_{j+\frac{1}{2}}^n + D_{j-\frac{1}{2}}^n \leq 1. \tag{A.8}$$

Hence by (A.6) and (A.8) inductively

$$
\begin{aligned}
|u_j^{n+1}| &\leq (1 - C_{j+\frac{1}{2}}^n - D_{j-\frac{1}{2}}^n)|u_j^n| + C_{j+\frac{1}{2}}^n |u_{j+1}^n| + D_{j-\frac{1}{2}}^n |u_{j-1}^n| \\
&\leq (1 - C_{j+\frac{1}{2}}^n - D_{j-\frac{1}{2}}^n) \max\{|u_{j-1}^n|, |u_j^n|, |u_{j+1}^n|\} \\
&+ C_{j+\frac{1}{2}}^n \max\{|u_{j-1}^n|, |u_j^n|, |u_{j+1}^n|\} + D_{j-\frac{1}{2}}^n \max\{|u_{j-1}^n|, |u_j^n|, |u_{j+1}^n|\} \\
&= \max\{|u_{j-1}^n|, |u_j^n|, |u_{j+1}^n|\} \\
&\leq \cdots \leq \sup_j |u_j^0| \leq \|u^0\|_{L^\infty(\mathbb{R})}.
\end{aligned}
$$

This means that the CFL-condition (A.3) will be satisfied in every time step if it is initially satisfied and it is justified to choose the same timestep $\Delta t = c\Delta x$ for all n.

Bound on the total variation

We consider again the scheme in the incremental form (A.6). We have

$$
\begin{aligned}
|1 - C_{j+\frac{1}{2}}^n - D_{j+\frac{1}{2}}^n| &= \frac{\Delta t}{2\Delta x} |\overline{u}_j^n - \overline{u}_{j+1}^n| \\
&\leq \frac{\Delta t}{2\Delta x} \frac{2\Delta x}{\alpha} \left| \sum_l \left(g\left(\frac{x_j - x_l}{\alpha}\right) - g\left(\frac{x_{j+1} - x_l}{\alpha}\right)\right) u_l^n \right| \\
&\leq \frac{\Delta t}{\alpha} \sum_l \left| g\left(\frac{x_j - x_l}{\alpha}\right) - g\left(\frac{x_{j+1} - x_l}{\alpha}\right)\right| \sup_k |u_k^n| \\
&\leq \frac{\Delta t}{\alpha} \sum_l \left| g\left(\frac{x_l}{\alpha}\right) - g\left(\frac{x_l + \Delta x}{\alpha}\right)\right| \sup_k |u_k^n| \\
&\leq \frac{\Delta t}{\alpha} \mathrm{TV}(g) \sup_k |u_k^n| \\
&\leq \frac{\Delta t \, C_g}{\alpha} \sup_k |u_k^0|
\end{aligned}
\tag{A.9}
$$

using the beforehand proved bound on the maximum. We rewrite the difference $u_{j+1}^{n+1} - u_j^{n+1}$ in terms of the incremental coefficients

$$
\begin{aligned}
u_{j+1}^{n+1} - u_j^{n+1} &= \left(1 - C_{j+\frac{1}{2}}^n - D_{j+\frac{1}{2}}^n\right)(u_{j+1}^n - u_j^n) \\
&+ C_{j+\frac{3}{2}}^n (u_{j+2}^n - u_{j+1}^n) + D_{j-\frac{1}{2}}^n (u_j^n - u_{j-1}^n).
\end{aligned}
\tag{A.10}
$$

Hence

$$
\begin{aligned}
\mathrm{TV}(u^{n+1}) &= \sum_j |u_{j+1}^{n+1} - u_j^{n+1}| \\
&= \sum_j |(1 - C_{j+\frac{1}{2}}^n - D_{j+\frac{1}{2}}^n)(u_{j+1}^n - u_j^n) \\
&\quad + C_{j+\frac{3}{2}}^n (u_{j+2}^n - u_{j+1}^n) + D_{j-\frac{1}{2}}^n (u_j^n - u_{j-1}^n)| \\
&\leq \sum_j \big(|1 - C_{j+\frac{1}{2}}^n - D_{j+\frac{1}{2}}^n|\,|u_{j+1}^n - u_j^n| \\
&\quad + C_{j+\frac{3}{2}}^n |u_{j+2}^n - u_{j+1}^n| + D_{j-\frac{1}{2}}^n |u_j^n - u_{j-1}^n|\big) \\
&\leq \sum_j \big(|1 - C_{j+\frac{1}{2}}^n - D_{j+\frac{1}{2}}^n| + C_{j+\frac{1}{2}}^n + D_{j+\frac{1}{2}}^n\big)|u_{j+1}^n - u_j^n| \\
&= \sum_j \left(\frac{\Delta t}{2\Delta x}\big(|\overline{u}_j^n - \overline{u}_{j+1}^n| - \overline{u}_j^n + \overline{u}_{j+1}^n\big) + 1\right)|u_{j+1}^n - u_j^n| \\
&\leq \sum_j \left(\frac{2\Delta t\, C_g}{\alpha} \sup_k |u_k^0| + 1\right)|u_{j+1}^n - u_j^n| \\
&= \mathrm{TV}(u^n) \cdot \left(\frac{2\Delta t\, C_g}{\alpha} \sup_k |u_k^0| + 1\right)
\end{aligned}
$$

using (A.9). Iterating over n, we obtain

$$
\begin{aligned}
\mathrm{TV}(u^n) &\leq \left(\frac{2\Delta t\, C_g}{\alpha} \sup_k |u_k^0| + 1\right)^n \cdot \mathrm{TV}(u^0) \\
&\leq \exp\left\{\frac{2\Delta t\, C_g}{\alpha} \sup_k |u_k^0| \cdot n\right\} \cdot \mathrm{TV}(u^0) \\
&\leq \exp\left\{\frac{C_{g,u^0}\, T}{\alpha}\right\} \cdot TV(u^0) \\
&\leq C_{\alpha,u^0,T}\, \mathrm{TV}(u^0), \qquad \text{(A.11)}
\end{aligned}
$$

where we used $(1 + x) \leq e^x$, $x \in \mathbb{R}$, in the second inequality.

Bounds on the first and second differences

Bound on the first difference $u_{j+1}^n - u_j^n$*:*
We start by bounding the first difference $u_{j+1}^n - u_j^n$ in terms of the maximum

of the first derivative of the initial data. By (A.7), (A.9) and (A.10) we have

$$
\begin{aligned}
|u^{n+1}_{j+1} - u^{n+1}_j| \leq\; & \frac{2\Delta t\, C_g}{\alpha} \sup_k |u^0_k|\, |u^n_{j+1} - u^n_j| \\
& + C^n_{j+\frac{3}{2}} |u^n_{j+2} - u^n_{j+1}| + D^n_{j-\frac{1}{2}} |u^n_j - u^n_{j-1}| \\
\leq\; & \left(\frac{2\Delta t\, C_g}{\alpha} \sup_k |u^0_k| + 1 + \frac{\Delta t}{2\Delta x} (\overline{u}^n_{j+1} - \overline{u}^n_j) \right) \\
& \cdot \max\{ |u^n_{j+2} - u^n_{j+1}|, |u^n_{j+1} - u^n_j|, |u^n_j - u^n_{j-1}| \} \\
\leq\; & \left(\frac{4\Delta t\, C_g}{\alpha} \sup_k |u^0_k| + 1 \right) \\
& \cdot \max\{ |u^n_{j+2} - u^n_{j+1}|, |u^n_{j+1} - u^n_j|, |u^n_j - u^n_{j-1}| \}.
\end{aligned}
$$

We iterate over n and use $(1 + x) \leq e^x$, $x \in \mathbb{R}$, in the same way as in the proof of the TV-bound to obtain

$$
\begin{aligned}
|u^n_{j+1} - u^n_j| \leq\; & \left(\frac{4\Delta t\, C_g}{\alpha} \sup_k |u^0_k| + 1 \right)^n \cdot \sup_k |u^0_{k+1} - u^0_k| \\
\leq\; & \exp\left(\frac{C_{g,u^0}\, T}{\alpha} \right) \cdot \sup_k |u^0_{k+1} - u^0_k| \\
\leq\; & C_{\alpha,u^0,T}\, \Delta x \|u^0\|_{C^1}
\end{aligned}
\tag{A.12}
$$

where $\|u^0\|_{C^1} = \max\{\sup_{x\in\mathbb{R}} |u^0(x)|, \sup_{x\in\mathbb{R}} |(u^0)'(x)|\}$.

Estimates on $|\overline{u}^n_{j+1} - \overline{u}^n_j|$*:*
In a second step, we bound the difference of the approximation of the filtered velocity. We can estimate $|\overline{u}^n_{j+1} - \overline{u}^n_j|$ in different ways. We can use

$$
\begin{aligned}
|\overline{u}^n_{j+1} - \overline{u}^n_j| =\; & \left| \Delta x \sum_l g_\alpha(x_{j+1} - x_l) - g_\alpha(x_j - x_l)) u^n_l \right| \\
\leq\; & \Delta x \sum_l |g_\alpha(x_{j+1} - x_l) - g_\alpha(x_j - x_l)| \sup_k |u^n_k| \\
=\; & \Delta x\, \mathrm{TV}(g_\alpha) \sup_k |u^n_k| \\
=\; & \frac{\Delta x}{\alpha} \mathrm{TV}(g) \sup_k |u^n_k|
\end{aligned}
\tag{A.13}
$$

or

$$
\begin{aligned}
|\overline{u}^n_{j+1} - \overline{u}^n_j| =\; & \left| \Delta x \sum_l (g_\alpha(x_l)(u^n_{j+1-l} - u^n_{j-l}) \right| \\
\leq\; & \Delta x\, |g_\alpha(0)| \sum_l |u^n_{j+1-l} - u^n_{j-l}| \\
=\; & \frac{\Delta x}{\alpha} |g(0)|\, \mathrm{TV}(u^n)
\end{aligned}
\tag{A.14}
$$

or

$$
\begin{aligned}
|\overline{u}_{j+1}^n - \overline{u}_j^n| &= \left|\Delta x \sum_l (g_\alpha(x_l)(u_{j+1-l}^n - u_{j-l}^n)\right| \\
&\leq \Delta x \sum_l |g_\alpha(x_l)| \sup_k |u_{k+1}^n - u_k^n| \\
&\leq \|g\|_{L^1(\mathbb{R})} \sup_k |u_{k+1}^n - u_k^n| \qquad \text{(A.15)}
\end{aligned}
$$

to bound the first difference of the filtered velocity. Estimates (A.13), (A.14) and (A.15) are independent of the numerical scheme used to discretize the first equation in (A.1).

Bound on the second difference $(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$*:*
Now we are ready to bound the second difference $(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$. We rewrite it in terms of our finite difference scheme

$$
\begin{aligned}
u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} &= \frac{1}{2}\left(u_{j+2}^n - 2u_{j+1}^n + u_j^n + u_j^n - 2u_{j-1}^n + u_{j-2}^n\right) \\
&- \frac{\Delta t}{2\Delta x}\left\{\overline{u}_{j+1}^n(u_{j+2}^n - u_j^n) - 2\overline{u}_j^n(u_{j+1}^n - u_{j-1}^n) + \overline{u}_{j-1}^n(u_j^n - u_{j-2}^n)\right\}. \qquad \text{(A.16)}
\end{aligned}
$$

Adding and subtracting terms, the second part on the right hand side of (A.16) can be rewritten as

$$
\begin{aligned}
&\overline{u}_{j+1}^n(u_{j+2}^n - u_j^n) - 2\overline{u}_j^n(u_{j+1}^n - u_{j-1}^n) + \overline{u}_{j-1}^n(u_j^n - u_{j-2}^n) \\
&= (\overline{u}_{j+1}^n - \overline{u}_{j-1}^n)\left(\frac{u_{j+2}^n - 2u_{j+1}^n + u_j^n}{2} + u_{j+1}^n - 2u_j^n + u_{j-1}^n + \frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{2}\right) \\
&\quad + \left(\frac{\overline{u}_{j+1}^n + \overline{u}_{j-1}^n}{2}\right)\left((u_{j+2}^n - 2u_{j+1}^n + u_j^n) - (u_j^n - 2u_{j-1}^n + u_{j-2}^n)\right) \\
&\quad + (\overline{u}_{j+1}^n - 2\overline{u}_j^n + \overline{u}_{j-1}^n)(u_{j+1}^n - u_{j-1}^n) \qquad \text{(A.17)}
\end{aligned}
$$

On the right hand side of (A.17) we recognize a discrete version of $\overline{u}_x u_{xx}$, $\overline{u} u_{xxx}$ and $\overline{u}_{xx} u_x$. We estimate $(\overline{u}_{j+1}^n - 2\overline{u}_j^n + \overline{u}_{j-1}^n)$:

$$
\begin{aligned}
|\overline{u}_{j+1}^n - 2\overline{u}_j^n + \overline{u}_{j-1}^n| &= \left|\Delta x \sum_l g^\alpha(x_l)(u_{j+1-l}^n - 2u_{j-l}^n + u_{j-1-l}^n)\right| \\
&\leq \Delta x \sum_l |g^\alpha(x_l)| \sup_k |u_{k+1}^n - 2u_k^n + u_{k-1}^n| \\
&\leq \|g\|_{L^1(\mathbb{R})} \sup_k |u_{k+1}^n - 2u_k^n + u_{k-1}^n|
\end{aligned}
$$

Thus we can bound the third term on the right-hand side of (A.17),

$$
\begin{aligned}
&|(\overline{u}_{j+1}^n - 2\overline{u}_j^n + \overline{u}_{j-1}^n)(u_{j+1}^n - u_{j-1}^n)| \\
&\qquad \leq |u_{j+1}^n - u_{j-1}^n| \|g\|_{L^1(\mathbb{R})} \sup_k |u_{k+1}^n - 2u_k^n + u_{k-1}^n| \\
&\qquad \leq C_{\alpha,u^0,T}\, \Delta x \|u^0\|_{C^1} \|g\|_{L^1(\mathbb{R})} \sup_k |u_{k+1}^n - 2u_k^n + u_{k-1}^n|, \qquad \text{(A.18)}
\end{aligned}
$$

where we used the bound on the first difference, (A.12) in the second inequality. For the first term on the right-hand side of (A.17), we have by (A.13)

$$\begin{aligned}(\overline{u}^n_{j+1}-\overline{u}^n_{j+1})&\left(\frac{u^n_{j+2}-2u^n_{j+1}+u^n_j}{2}+u^n_{j+1}-2u^n_j+u^n_{j-1}+\frac{u^n_j-2u^n_{j-1}+u^n_{j-2}}{2}\right)\\ &\leq \frac{\Delta x}{\alpha}\operatorname{TV}(g)\sup_k|u^n_k|\sup_k|u^n_{k+1}-2u^n_k+u^n_{k-1}|\\ &\leq \frac{\Delta x}{\alpha}\operatorname{TV}(g)\,\|u^0\|_{L^\infty}\sup_k|u^n_{k+1}-2u^n_k+u^n_{k-1}| \qquad (A.19)\end{aligned}$$

So, what is left to bound on the right-hand side of (A.16) is the term

$$\begin{aligned}A :=\ &\frac{1}{2}\left(u^n_{j+2}-2u^n_{j+1}+u^n_j+u^n_j-2u^n_{j-1}+u^n_{j-2}\right)\\ &-\frac{\Delta t}{2\Delta x}\left(\frac{\overline{u}^n_{j+1}+\overline{u}^n_{j-1}}{2}\right)\left((u^n_{j+2}-2u^n_{j+1}+u^n_j)-(u^n_j-2u^n_{j-1}+u^n_{j-2})\right)\\ &=\left(\frac{1}{2}-\frac{\Delta t}{2\Delta x}\left(\frac{\overline{u}^n_{j+1}+\overline{u}^n_{j-1}}{2}\right)\right)(u^n_{j+2}-2u^n_{j+1}+u^n_j)\\ &\quad+\left(\frac{1}{2}+\frac{\Delta t}{2\Delta x}\left(\frac{\overline{u}^n_{j+1}+\overline{u}^n_{j-1}}{2}\right)\right)(u^n_j-2u^n_{j-1}+u^n_{j-2}) \qquad (A.20)\end{aligned}$$

Thanks to the CFL-condition (A.3),

$$\frac{1}{2}\pm\frac{\Delta t}{2\Delta x}\frac{\overline{u}^n_{j+1}+\overline{u}^n_{j-1}}{2}\geq 0$$

and we can take absolute values in Equation (A.20) and the supremum over all $k\in\mathbb{N}$ to obtain

$$|A|\leq\sup_k|u^n_{k+1}-2u^n_k+u^n_{k-1}| \qquad (A.21)$$

Combining (A.18), (A.19) and (A.21), we get

$$\begin{aligned}&|u^{n+1}_{j+1}-2u^{n+1}_j+u^{n+1}_{j-1}|\\ &\leq\left(1+\frac{\Delta t}{2}\left(\|g\|_{L^1(\mathbb{R})}\,C_{\alpha,u^0,T}\,\|u^0\|_{C^1}+\frac{2}{\alpha}\operatorname{TV}(g)\,\|u^0\|_{L^\infty}\right)\right)\sup_k|u^n_{k+1}-2u^n_k+u^n_{k-1}|\\ &\qquad\leq\left(1+\frac{\Delta t}{2}C_{\alpha,u^0,T,g}\right)\sup_k|u^n_{k+1}-2u^n_k+u^n_{k-1}|\end{aligned}$$

Using induction over n and $(1+x)\leq e^x$, $x\in\mathbb{R}$, in the same way as we did in order to bound the total variation and the first difference in (A.12), we obtain

$$\begin{aligned}|u^n_{j+1}-2u^n_j+u^n_{j-1}|&\leq\exp\left(C_{\alpha,u^0,T,g}\,T\right)\cdot\sup_k|u^0_{k+1}-2u^0_k+u^0_{k-1}|\\ &\leq\widetilde{C}_{\alpha,u^0,T,g}\Delta x^2\|u^0\|_{C^2} \qquad (A.22)\end{aligned}$$

First order accuracy

Truncation error:
We denote by $\widetilde{u}_j^n := u^\alpha(x_j, t^n)$ the exact solution of (A.1) at $(x,t) = (x_j, t^n)$, $j \in \mathbb{N}$, $n = 0, \dots, N$; and by $H_j^{n-m}(\widetilde{u}^m)$, $0 \leq m \leq n$ the approximation computed at $(x,t) = (x_j, t^n)$ by $n-m$ steps with Scheme (A.4) from the exact solution at time $t = t^m$. Then the local truncation error of the scheme is defined as

$$\tau_{j,n} := \frac{1}{\Delta t}\Big(\widetilde{u}_j^{n+1} - H_j(\widetilde{u}^n)\Big).$$

We denote

$$\widehat{u}_j^n := (g_\alpha * u^\alpha)(x_j, t^n)$$

the exact convolution of the filter g_α with u^α and

$$\overline{\widetilde{u}}_j^n := \frac{\Delta x}{\alpha} \sum_{l \in \mathbb{N}} g\left(\frac{x_j - x_l}{\alpha}\right) \widetilde{u}_l^n.$$

the approximated convolution. Then, since by Taylor's Theorem

$$\begin{aligned}
\int_{x_{l-1/2}}^{x_{l+1/2}} g_\alpha(x_j - y) u^\alpha(y)\, dy &= \int_{x_{l-1/2}}^{x_{l+1/2}} \Big(g_\alpha(x_j - x_l) u^\alpha(x_l) \\
&\quad + (y - x_l)(g_\alpha(x_j - x_l) u^\alpha(x_l))_x + \frac{(y-x_l)^2}{2}(g_\alpha(x_j - \xi_y) u^\alpha(\xi_y))_{xx}\Big)\, dy \\
&= \Delta x g_\alpha(x_j - x_l) u^\alpha(x_l) + \int_{x_{l-1/2}}^{x_{l+1/2}} \frac{(y-x_l)^2}{2}(g_\alpha(x_j - \xi_y) u^\alpha(\xi_y))_{xx}\, dy,
\end{aligned}$$

where $\xi_y \in [x_{l-1/2}, x_{l+1/2}]$, we have

$$\begin{aligned}
|\widehat{u}_j^n - \overline{\widetilde{u}}_j^n| &= \Big| \int g_\alpha(x_j - y) u^\alpha(y)\, dy - \frac{\Delta x}{\alpha} \sum_{l \in \mathbb{N}} g\left(\frac{x_j - x_l}{\alpha}\right) \widetilde{u}_l^n \Big| \\
&= \Big| \sum_l \Big(\int_{x_{l-1/2}}^{x_{l+1/2}} g_\alpha(x_j - y) u^\alpha(y)\, dy - \frac{\Delta x}{\alpha} g\left(\frac{x_j - x_l}{\alpha}\right) \widetilde{u}_l^n \Big) \Big| \\
&= \Big| \sum_l \int_{x_{l-1/2}}^{x_{l+1/2}} \frac{(y-x_l)^2}{2}(g_\alpha(x_j - \xi_y) u^\alpha(\xi_y))_{xx}\, dy \Big| \\
&\leq \sum_l \Big| \int_{x_{l-1/2}}^{x_{l+1/2}} \frac{(y-x_l)^2}{2}(g_{\alpha,xx}(x_j - \xi_l) u^\alpha(\xi_l) \\
&\qquad + g_{\alpha,x}(x_j - \xi_l) u_x^\alpha(\xi_l) + g_\alpha(x_j - \xi_l) u_{xx}^\alpha(\xi_l))\, dy \Big| \\
&\leq C_\alpha \|u^0\|_{C^2} \Delta x^2 \sum_l \int_{x_{l-1/2}}^{x_{l+1/2}} |g_{\alpha,xx}(x_j - \xi_l)| + |g_{\alpha,x}(x_j - \xi_l)| + |g_\alpha(x_j - \xi_l)| dy \\
&\leq \widetilde{C}_\alpha \|u^0\|_{C^2} \Delta x^2 \|g_\alpha\|_{W^{2,1}(\mathbb{R})} \\
&= C_{\alpha,g,T,u^0} \Delta x^2,
\end{aligned}$$

where we have used that u_x^α, u_{xx}^α are bounded in terms of the first and second derivative of the initial data (see Chapter 2) in the third inequality. Using this, Taylor expansion and that u^α satisfies Equation (A.1a), we can write $\widetilde{u}_j^{n+1}$ as

$$\begin{aligned}
\widetilde{u}_j^{n+1} &= \widetilde{u}_j^n + \Delta t\,(\widetilde{u}_j^n)_t + \frac{\Delta t^2}{2} u_{tt}^\alpha(x_j,\xi), \quad \xi \in [t^n, t^{n+1}] \\
&= \widetilde{u}_j^n - \Delta t\, \widehat{u}_j^n (\widetilde{u}_j^n)_x + \mathcal{O}(\Delta t^2) \\
&= \widetilde{u}_j^n - \Delta t\, \overline{\widetilde{u}}_j^n (\widetilde{u}_j^n)_x + \mathcal{O}(\Delta t^2, \Delta x^2 \Delta t\, u_x^\alpha) \\
&= \widetilde{u}_j^n - \Delta t\, \overline{\widetilde{u}}_j^n (\widetilde{u}_j^n)_x + \mathcal{O}(\Delta t \Delta x),
\end{aligned} \tag{A.23}$$

since u_x^α is bounded in terms of the derivative of u^0 and we can rewrite u_{tt}^α by differentiating Equation (A.1a),

$$\begin{aligned}
u_{tt}^\alpha &= -(\overline{u}^\alpha u_x^\alpha)_t \\
&= -\overline{u}_t^\alpha u_x^\alpha - \overline{u}^\alpha u_{xt}^\alpha \\
&= \overline{\overline{u}^\alpha u_x^\alpha} u_x^\alpha + \overline{u}^\alpha (\overline{u}^\alpha u_x^\alpha)_x,
\end{aligned}$$

which is bounded in terms of the first and second derivatives of the initial data u^0. For the approximation $H(\widetilde{u}_j^n)$ and some $\xi_1, \xi_2 \in [x_{j-1}, x_{j+1}]$, we have

$$\begin{aligned}
H_j(\widetilde{u}^n) &= \frac{\widetilde{u}_{j-1}^n + \widetilde{u}_{j+1}^n}{2} - \frac{\Delta t}{2\Delta x} \overline{\widetilde{u}}_j^n (\widetilde{u}_{j+1}^n - \widetilde{u}_{j-1}^n) \\
&= \widetilde{u}_j^n + \frac{\Delta x^2}{4} (u_{xx}^\alpha(\xi_1, t^n) + u_{xx}^\alpha(\xi_2, t^n)) \\
&\quad - \frac{\Delta t}{2\Delta x} \overline{\widetilde{u}}_j^n \left(2\Delta x (\widetilde{u}_j^n)_x + \frac{\Delta x^2}{2} (u_{xx}^\alpha(\xi_1, t^n) - u_{xx}^\alpha(\xi_2, t^n)) \right) \\
&= \widetilde{u}_j^n - \Delta t \overline{\widetilde{u}}_j^n (\widetilde{u}_j^n)_x + \mathcal{O}(\Delta x \Delta t),
\end{aligned} \tag{A.24}$$

again, because u_{xx}^α is bounded in terms of the first and second derivatives of the initial data u^0 and by the CFL-condition $\Delta t = c\Delta x$. We subtract (A.24) from (A.23) to obtain

$$\tau_{j,n} \leq C_\alpha \Delta x. \tag{A.25}$$

Hence our scheme is consistent.

Stability:
We denote $e_j^n = |\widetilde{u}_j^n - H_j^n(\widetilde{u}^0)|$ the error at time $t = t^n$ and $E^n = \sup_k e_k^n$.

Note that $e_j^n \leq 2\max\{|\widetilde{u}_j^n|, |H_j^n(\widetilde{u}^0)\} \leq 2\|u^0\|_{L^\infty}$ Then we have

$$
\begin{aligned}
&H_j(\widetilde{u}^n + e^n) - H_j(\widetilde{u}^n) \\
&= \frac{e_{j-1}^n + e_{j+1}^n}{2} - \frac{\Delta t}{2\Delta x}(\overline{\widetilde{u}}_j^n + \overline{e}_j^n)(\widetilde{u}_{j+1}^n + e_{j+1}^n - \widetilde{u}_{j-1}^n - e_{j-1}^n) + \frac{\Delta t}{2\Delta x}\overline{\widetilde{u}}_j^n(\widetilde{u}_{j+1}^n - \widetilde{u}_{j-1}^n) \\
&= \left(\frac{1}{2} - \frac{\Delta t}{2\Delta x}(\overline{\widetilde{u}}_j^n + \overline{e}_j^n)\right)e_{j+1}^n + \left(\frac{1}{2} + \frac{\Delta t}{2\Delta x}(\overline{\widetilde{u}}_j^n + \overline{e}_j^n)\right)e_{j-1}^n - \frac{\Delta t}{2\Delta x}\overline{e}_j^n(\widetilde{u}_{j+1}^n - \widetilde{u}_{j-1}^n) \\
&= \left(\frac{1}{2} - \frac{\Delta t}{2\Delta x}(\overline{\widetilde{u}}_j^n + \overline{e}_j^n)\right)e_{j+1}^n + \left(\frac{1}{2} + \frac{\Delta t}{2\Delta x}(\overline{\widetilde{u}}_j^n + \overline{e}_j^n)\right)e_{j-1}^n \\
&\quad - \frac{\Delta t}{2\Delta x}\overline{e}_j^n\left(2\Delta x(\widetilde{u}_j^n)_x + \frac{\Delta x^2}{2}(u_{xx}^\alpha(\xi_1, t^n) - u_{xx}^\alpha(\xi_2, t^n)\right),
\end{aligned}
$$

$\xi_1, \xi_2 \in [x_{j-1}, x_{j+1}]$. We use the CFL-condition, $\overline{e}_j^n \leq E^n$, and that the first and the second derivative of u^α are bounded in terms of the derivatives of the initial data, to estimate the last term by

$$
|H_j(\widetilde{u}^n + e^n,) - H_j(\widetilde{u}^n)| \leq \left(1 + C_{\alpha,u^0,T}\,\Delta x\|u^0\|_{C^1}\right)E^n. \tag{A.26}
$$

First order accuracy: We decompose the error at time t^n, use the estimate on the truncation error (A.25) and the stability estimate (A.26), to obtain

$$
\begin{aligned}
e_j^n &= |\widetilde{u}_j^n - H_j^n(\widetilde{u}^0)| \\
&\leq |\widetilde{u}_j^n - H_j(\widetilde{u}^{n-1})| + |H_j(\widetilde{u}^{n-1}) - H_j^n(\widetilde{u}^0)| \\
&\leq |\widetilde{u}_j^n - H_j(\widetilde{u}^{n-1})| + |H_j(\widetilde{u}^{n-1}) - H_j(\widetilde{u}^{n-1} + e^{n-1})| \\
&\leq C_1\Delta x\Delta t + \left(1 + C_2\,\Delta x\right)E^{n-1}.
\end{aligned}
$$

Thus

$$
E^n \leq C_1\Delta x\Delta t + \left(1 + C_2\,\Delta x\right)E^{n-1}. \tag{A.27}
$$

Using induction over the number of timesteps n, we compute

$$
\begin{aligned}
E^n &\leq C_1\Delta x\Delta t\sum_{m=0}^{n-1}\left(1 + C_2\,\Delta x\right)^m \\
&\leq C_1\Delta x\Delta t\frac{\left(1 + C_2\,\Delta x\right)^n - 1}{C_2\Delta x} \\
&\leq \frac{C_1\Delta x}{C_2}\exp\{nC_2\,\Delta x\} \\
&= C_3\Delta x\exp\{C_4 t^n\}
\end{aligned}
$$

where we used the CFL-condition $\Delta t = c\Delta x$ in the third inequality. So,

$$
E^N \leq \Delta x\exp\{C\,T\}
$$

where C is a constant depending on $\|u^0\|_{C^2}$, α, T and g. Since we are assuming that $u^0 \in C^2(\mathbb{R})$, we have for $x \in [x_{j-1/2}, x_{j+1/2}]$,

$$|u^\alpha(x,T) - u^\alpha(x_j,T)| \leq \Delta x\, C_{\alpha,T,g,u^0}$$

and thus finally

$$|u^{\Delta x}(x,T) - u^\alpha(x,T)| \leq \Delta x\, C_{\alpha,T,g,u^0}. \qquad \square$$

Remark A.2 *The bound on the total variation of the approximation at time T, (A.11), is useless in the limit $\alpha \to 0$. In spite of this, we know that the total variation of the solution to equations (A.1) at any time is bounded by the total variation of the initial data, uniformly in α. Numerical experiments indicate that the approximations computed with Scheme (A.4) might not share this property in the limit $\alpha \to 0$. In particular, we have tested the scheme for $\alpha = 0$ on the domain $[0, 2\pi)$ with periodic boundary conditions and the following type of initial data:*

$$u^0(x) = \chi_{((kx)\ mod\ (2\pi))<\pi}(x), \quad x \in [0, 2\pi), k \in \mathbb{N}, \tag{A.28}$$

where $c = a$ mod b is defined by $a = n \cdot b + c$, $n \in \mathbb{N}$, $|c| < |a|$ and $c > 0$. We choose the time step $\Delta t = 0.15\,\Delta x$, which satisfies the CFL-condition (A.3) and compute the approximation at time $T = 0.4$.

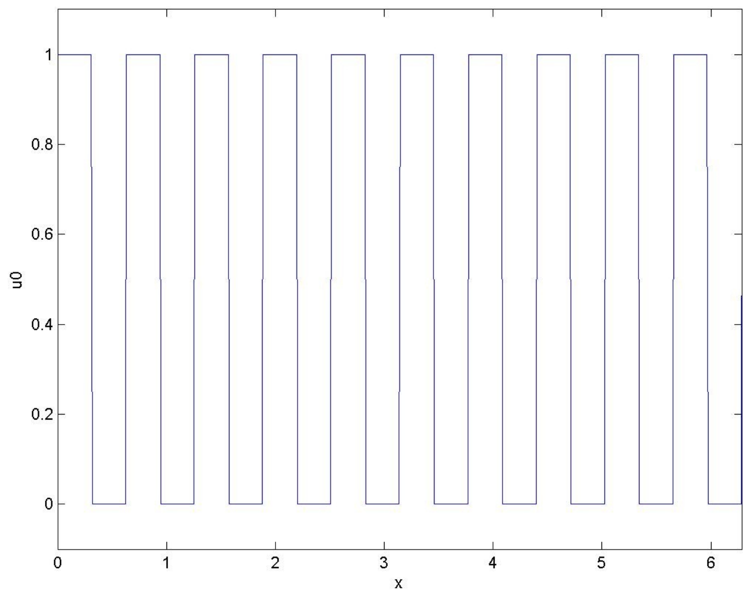

Figure A.1: The initial data (A.28) for $k = 10$ and 800 meshpoints in the interval $[0, 2\pi)$.

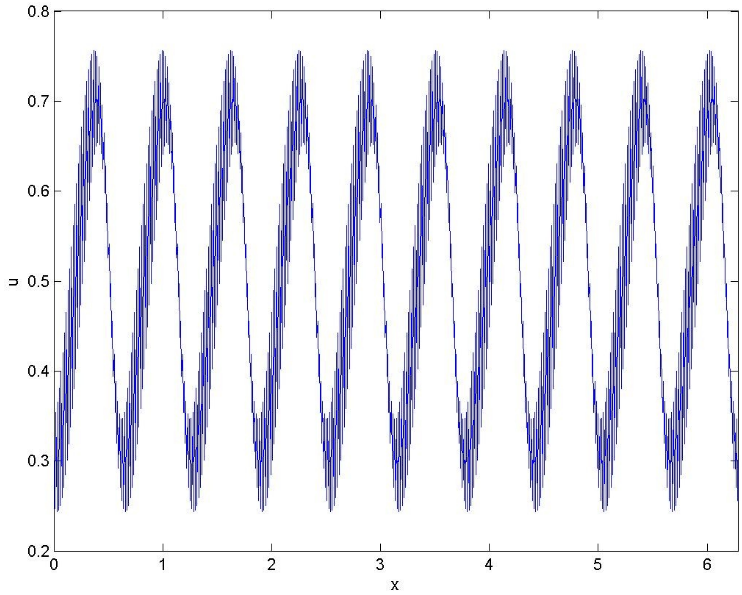

Figure A.2: Numerical approximation at time $T = 0.4$ computed with Scheme (A.4) for $\alpha = 0$ and the initial data (A.28), $k = 10$, 800 meshpoints.

$N_x \backslash k$	1	5	10	20
200	1.7005	1.5273	0.7637	1.0353
400	2.5831	3.0915	1.7893	0.7807
800	4.5593	6.0676	4.6425	1.8864
1600	9.579	13.006	12.227	6.0916

Table A.1: The ratio $\mathrm{TV}(u^N)/\mathrm{TV}(u^0)$ for the approximations of the initial value problem (A.28) by Scheme (A.4) for different k, different numbers of meshpoints N_x and fixed $T = 0.4$.

In Figure A.3, we see that the total variation of the approximation increases in time for $k = 10$ and 800 meshpoints. The initial data and the corresponding approximation can be seen in Figures A.1 and A.2.

If the approximation computed by Scheme (A.4) satisfied a bound

$$\sum_j |u_j^N - u_{j-1}^N| \leq \sum_j |u_j^0 - u_{j-1}^0|,$$

for all $\alpha > 0$, then it would also be satisfied for the limit case $\alpha = 0$. The numerical experiments show that this is not necessarily the case. The computed ratios in Table

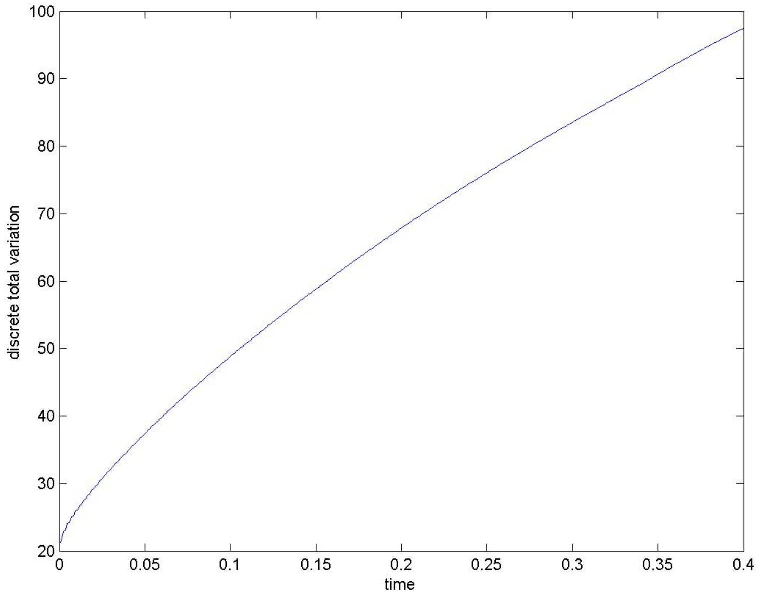

Figure A.3: Change of the total variation of the approximation by Scheme (A.4) to (A.28) with $k = 10$ in time computed on a mesh with 800 points.

A.2 indicate that maybe not even a bound of the form

$$\sum_j |u_j^N - u_{j-1}^N| \leq C_T \sum_j |u_j^0 - u_{j-1}^0|,$$

where C_T is a constant not depending on α and Δx, is available.

We could instead have used the scheme

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{\Delta t}{2\,\Delta x}\big(\overline{u}_{j+1/2}^n(u_{j+1}^n - u_j^n) + \overline{u}_{j-1/2}^n(u_j^n - u_{j-1}^n)\big), \tag{A.29a}$$

$$\overline{u}_{j\pm 1/2}^n = \frac{\Delta x}{\alpha} \sum_{l \in \mathbb{N}} g\left(\frac{x_{j\pm 1/2} - x_l}{\alpha}\right) u_l^n. \tag{A.29b}$$

$j \in \mathbb{Z}$, $n = 0, \dots, N$, which can be shown to be total variation diminishing and mass conserving independently of α. However, the bound on the maximum norm is depending on the parameter α. For fixed α, the approximations computed by this scheme converge to the solution of (A.1) as $\Delta x \to 0$, which can be shown in a similar way as it was done for Scheme (A.4) in the proof of Proposition A.1.

Bibliography

[1] H. S. Bhat and R. C. Fetecau. A Hamiltonian Regularization of the Burgers Equation. *Journal of Nonlinear Science*, 16:615–638, 2006.

[2] H. S. Bhat and R. C. Fetecau. The Riemann problem for the Leray-Burgers equation. *Journal of Differential Equations*, 246:3957–3979, 2009.

[3] H. S. Bhat, R. C. Fetecau, and J. Goodman. A Leray-type regularization for the isentropic Euler equations. *Nonlinearity*, 20:2035–2046, 2007.

[4] V. I. Bogachev. *Measure Theory*. Springer-Verlag, 2007.

[5] A. Cheskidov, D. D. Holm, E. Olson, and E. S. Titi. On a Leray alpha model of turbulence. In *Proceedings of the Royal Society A*, volume 461, pages 629–649, 2005.

[6] G. M. Coclite and K. H. Karlsen. Hamiltonian Approximation of Entropy Solutions of the Burgers Equation. In *Proceedings of HYP 2010*, to appear.

[7] G. M. Coclite, K. H. Karlsen, S. Mishra, and N. H. Risebro. A hyperbolic-elliptic model of two-pase flow in porous media – Existence of entropy solutions. SAM Report 6, ETH Zürich, February 2011.

[8] J. W. Deardorff. A numerical study of three-dimensional turbulent channel flow at large Reynolds numbers. *Journal of Fluid Mechanics*, 41:453–480, 1970.

[9] C. R. Doering and J. D. Gibbon. *Applied Analysis of the Navier-Stokes Equations*. Cambridge University Press, 1995.

[10] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, 2002.

[11] U. S. Fjordholm and S. Mishra. Accurate numerical discretizations of non-conservative hyperbolic systems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 46:187–206, 2012.

[12] H. Holden and N. H. Risebro. *Front Tracking for Hyperbolic Conservation Laws*. Springer, 2002.

[13] P. D. Lax and C. D. Levermore. The Small Dispersion Limit of the Korteweg-de Vries Equation. I. *Communications on Pure and Applied Mathematics*, 36:253–290, 1983.

[14] P. D. Lax and C. D. Levermore. The Small Dispersion Limit of the Korteweg-de Vries Equation. II. *Communications on Pure and Applied Mathematics*, 36:571–593, 1983.

[15] L. Leray. Essai sur le mouvement d'un fluide visqueux emplissant l'espace. *Acta Mathematica*, 63:193–248, 1934.

[16] A. J. Majda and A. L. Bertozzi. *Vorticity and incompressible flow*. Cambridge University Press, 2002.

[17] K. Mohseni, B. Kosovic, S. Shkoller, and Jerrold E. Marsden. Numerical simulations of the Lagrangian averaged Navier-Stokes equations for homogeneous isotropic turbulence. *Physics of Fluids*, 15:524–544, 2003.

[18] G. Norgard and K. Mohseni. A new potential regularization of the one-dimensional Euler and homentropic Euler equations. *Multiscale Modeling and Simulation*, 8:1212–1243, 2010.

[19] G. J. Norgard and K. Mohseni. An examination of the homentropic Euler equations with averaged characteristics. *Journal of Differential Equations*, 248:574–593, 2010.

[20] G. J. Norgard and K. Mohseni. On the Convergence of the Convectively Filtered Burgers Equation to the Entropy Solution of the Inviscid Burgers Equation. *Multiscale Modeling & Simulation*, 7:1811–1837, 2010.

[21] J. C. Saut and R. Teman. Remarks on the Korteweg-de Vries equation. *Israel Journal of Mathematics*, 24(1):78–87, 1976.

[22] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*. Springer-Verlag, 1983.

[23] R. J. LeVeque Smoller. *Numerical methods for conservation laws*. Birkhäuser, 1990.

[24] Lloyd N. Trefethen. *Spectral Methods in Matlab*. Society for Industrial and Applied Mathematics, 2000.