Interface approximation and error estimates via mapping techniques

Varsakelis Christos

Supervisor: *Professor Dr Ralf Hiptmair*
Seminar of Applied Mathematics
Department of Mathematics
ETH Zurich

Abstract

In this paper we study the 2-d interface elliptic problem. Mapping techniques are used to obtain a perturbed problem, that work for finite elements of arbitrary order. Error estimates are discussed in the energy norm for linear Lagrangian finite elements. Optimal rates of convergence, with respect to meshwidth h, are recovered.

1 Motivation and methodology

Second order elliptic problems with discontinuous coefficients are common in Fluid Mechanics and Material Sciences. These problems, known in literature as interface problems, enjoy several properties that complicate numerical analysis, compared to usual elliptic problems. Their most important feature, is the low global regularity of the solution, even in the case where the interface is smooth enough [12-14]. In such cases the regularity of the solution is higher in the parts of the domain's partition, but globally this regularity is lost. A typical example is the following:
Assume that the data of the problem $f \in L^2(\Omega)$. Then locally the solution is in $H^2(\Omega)$ however, globally the function would be in $H^1(\Omega)$, where H^i stands for the classical Sobolev space.

This characteristic in combination with the irregular, in general, shape of the interface make the approximation with finite elements difficult. However, due to the already mentioned connection with Fluids and Materials several discussions have appeared in literature in the past few years [5-9,11,15]. The list presented here, cannot be considered in any case to be complete. Therefore, several methods have been used, both for conforming and non conforming

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finite elements. Some representative works are those of Babuska [4], where the interface problem is transformed into an equivalent minimization one, Chen-Zou [2] where the authors introduce some artificial Neumman conditions along the interface to secure H^1-compatibility and recover optimal convergence both in L^2-norm, up to a factor of $|log(h)|$, and H^1 norm. Moreover, Huang-Zou [3]discuss a non-conforming method, mortar finite elements and F.O.S.L.S., for the same interface problem which also leads to optimal convergence.

In this paper we propose a new conforming method based on mapping techniques for the interface problem. A piecewise diffeomorphism Φ is defined, mapping linear elements to curvilinear ones. The original problem is then transformed into a perturbed form. The error analysis naturally leads to the application of first Strang's lemma. Our method, inspired by [1], is presented here for linear Lagrangian finite elements in 2-d, however, it can extended to every order.

1.1 The interface problem

Let $\Omega \subset \mathbb{R}^2$, a convex polygon and Γ be a C^2 curve creating a partition of Ω into Ω_1, Ω_2 as it is displayed in Figure 1. We assume that Γ is given by a finite number of local charts, $\phi_i : D_i \subset \mathbb{R} \rightarrow \mathbb{R}^2$.

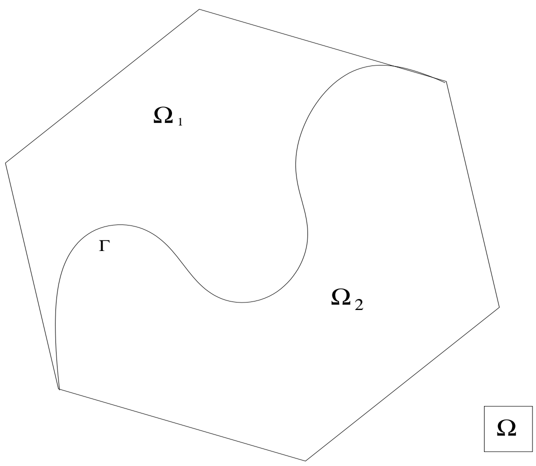

Fig. 1. The calculation domain

We consider the following problem

$$
\begin{aligned}
-\mathbf{div}(c(x)\nabla(u)) &= f, \text{ in } \Omega && (1.1)\\
u &= 0, \text{ in } \partial\Omega && (1.2)
\end{aligned}
$$

where

$$c(x) = \begin{cases} c_1, & \text{if } x \in \Omega_1 \\ c_2, & \text{if } x \in \Omega_2 \end{cases} \quad c_1, c_2 \in \mathbb{R}^+$$

The weak formulation of the problem reads:
Find $u \in H^1$ such as:

$$\int_\Omega c(x)\nabla(u) \cdot \nabla(v) = \int_\Omega fv \ \ \forall v \in H_0^1 \tag{1.3}$$

Note 1. For simplicity, we restrict ourselves to a piecewise constant positive conductivity tensor $c(x)$. In the general 2-d case, $c(x)$ is a a symmetric, uniformly positive definite function $C(x) \in L^\infty(\Omega, \mathbb{R}^{2,2})$. In this case we have to take into account the L^∞-norm of the conductivity tensor, hoewver, the proofs carry over. Moreover, we have not assumed anything for the data f. The restrictions for f will be introduced in the error analysis section, rising as natural requirements for the recovering of the optimal rate of convergence. However, we will begin our study by the usual assumption $f \in L^2$, in order to get the piecewise H^2-regularity of the solution.

We do Galerkin discretization under the following assumptions

- The triangulation of Ω T_h is obtained as follows:

(1) We approximate the interface via linear approximation as in figure 2
(2) Starting from the endpoints of the interface linear approximation we construct a regular triangulation of Ω, with meshwidth h, denoted by $\mathcal{M}$.

- We use linear Lagrangian finite elements with corresponding spaces

$$S_1^0(\Omega) = \{v \in C^0(\bar{\Omega}) : v_{|K} \in \mathcal{P}_1(K), , \forall K \in \mathcal{M}\}$$

Let now u_h denote the discrete solution obtained by Galerkin discretization. Our aim is to measure this error in the energy norm. This is going to be done in four steps

(1) We will define the piecewise diffeomorphism Φ that maps the elements which intersect the interface to curvilinear ones
(2) As already mentioned this will lead to the formulation of a perturbed discrete problem. We will do error analysis in this problem.
(3) We will estimate the error between the first and the second discrete solutions.
(4) We will obtain the final estimates through triangle inequality.

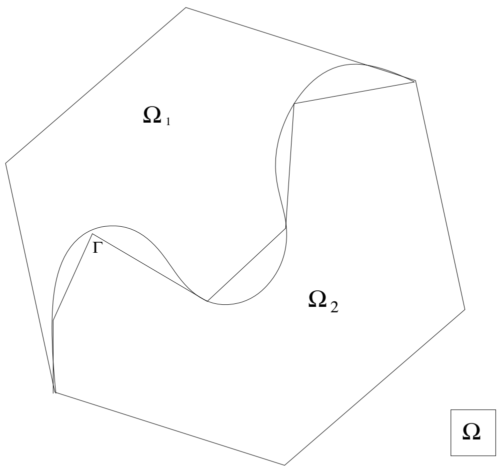

Fig. 2. Piecewise linear interpolation of the interface

2 The mapping Φ

We first assume that T_h consists of first order isoparametric finite elements i.e. the order of the elements in the domain of the triangulation is the same with the order of the reference element. This assumption will simplify much the calculations.

Φ must satisfy the following properties:

- It must be a local mapping i.e its value in an element must be depended only on points in this element
- It must be the identity map to elements that not have a face in the interface approximation
- The ϕ_i will play a crucial role in the recovering of the exact interface, as the next figure demonstrates. where Λ_1^1 is the usual interpolation operator of order 1

Under this assumptions and following [1] we have

Definition 2.1. We define the continuous mapping

$$\Phi : \Omega \to \Omega$$

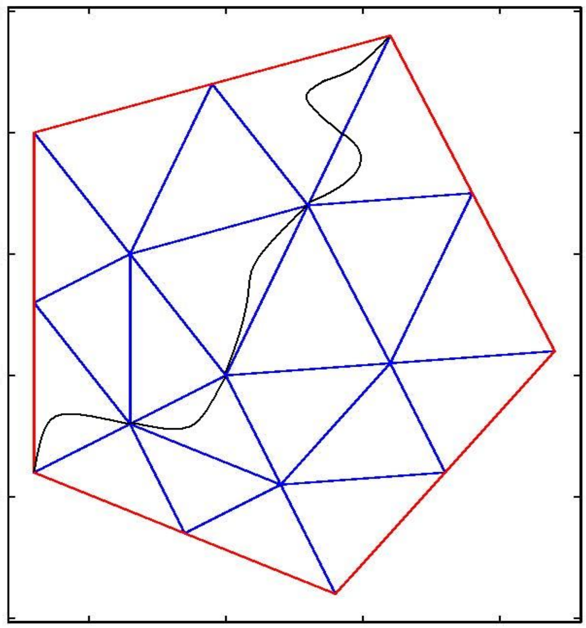

Fig. 3. Example of triangulation in a polygon with an interface after piecewise linear interpolation

through its restrictions to each element

$$\Psi_K : K \to \Psi(K)$$

as follows:

(1) If no face of K belongs to the interface approximation, we set $\Psi_K = I$
(2) Else, if one face of K, say $\gamma^1 K$ belongs to the interface approximation, we set

$$(\Psi_K - I) \circ F_K^1 = (1 - \lambda_3)^2 (\phi_i - \Lambda_1^1(\phi_i)) \circ G_{\hat{k}}^1 \circ \mathcal{F} \circ Z^p \tag{2.1}$$

- F_K^1 corresponds to the mapping that constructs 2-d isoparametric elements of order 1 from reference element $\hat{K}$
- λ_3 is the third barycentric coordinate of the point value
- ϕ_i is the corresponding local chart
- Λ_1^1 is the linear 1-d interpolation operator
- $G_{\hat{k}}^1$ is the mapping that constructs 1-d isoparametric elements of order 1 from reference element $\hat{k}$
- $\mathcal{F}$ is the natural isomorphism between a face of $\hat{K}$ and the 1-d reference element
- Z^p is the mapping that maps a point x of the reference element $\hat{K}$ to a point x' that belongs to a face of $\hat{K}$

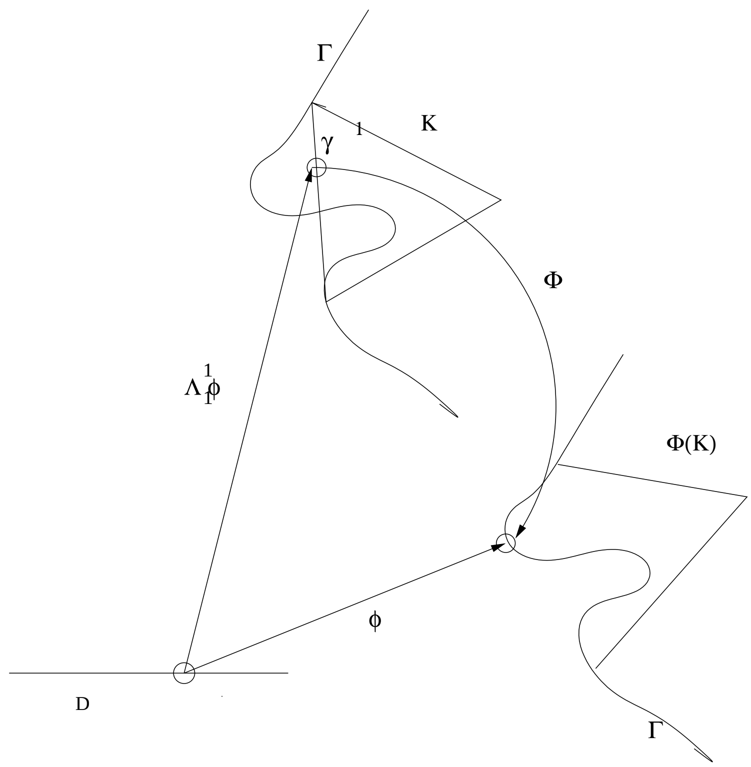

Fig. 4. Obtaining the exact interface approximation through ϕ_i

All the components of (2.1) are analyzed in appendix A, where Φ is constructed step by step explaining also the motivation for the various choices. Figure 5 shows the operation of Φ to an element that lies in the domain of the chart.

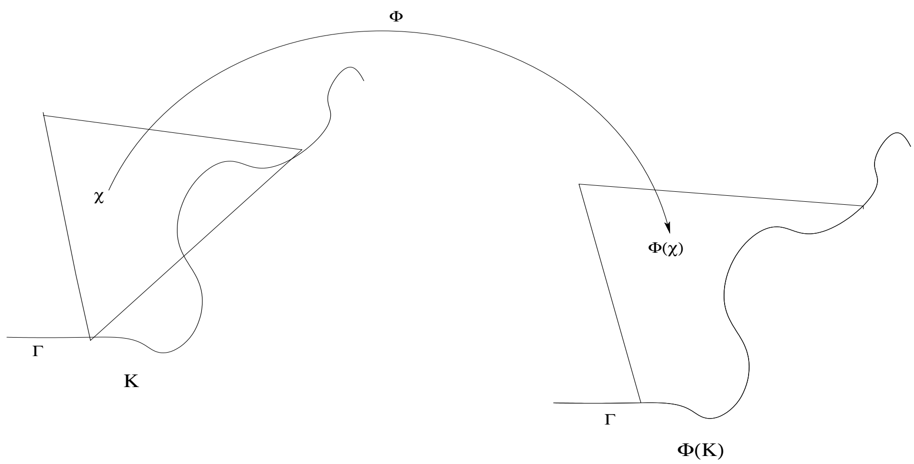

Fig. 5. The operation of Φ to an element in the domain of the chart

For our purposes we present, in the form of a proposition, the properties of Φ that we are going to use in our further error analysis. The proofs can be found in [1].

Proposition 2.2. *(1) There exists β depending only on the order of the*

Frechet derivative such as

$$\|D^s((\Psi_K - I))\|_{L^\infty(K)} \leq \beta(s)h^{2-s} \;\; \forall s \leq 2 \tag{2.2}$$

(2) The mappings Ψ_K are C^2 diffeomorphisms $K \to \Psi_K(K)$
(3) There exists γ such as

$$\sup_{x\in K} |J(\Psi_K)(x) - 1| \leq \gamma h \tag{2.3}$$

(4) The mapping $(\Psi_K)^{-1}$ satisfies

- *There exists $\delta(s)$ such as*

$$\|D^s((\Psi_K)^{-1} - I)\|_{L^\infty(K)} \leq \delta(s)h^{2-s} \;\; \forall s \leq 2 \tag{2.4}$$

- *there exists δ such as*

$$\sup_{x\in K} |J((\Psi_K)^{-1})(x) - 1| \leq \delta h \tag{2.5}$$

where $D^s(\Phi)$ stands for the Frechet derivative of order s, while $J\Phi$ corresponds to the determinant of the Jacobian of Φ

3 The perturbed problem and error estimates

Having defined Φ the perturbed problem arises naturally via transformation through Φ. The transformation of the eeliptic problem in our case reads

Find $u \circ \Phi \in H_0^1(\Omega)$ such as

$$\int_\Omega (c(x) \circ \Phi)(D\Phi)^{-T}\nabla(u \circ \Phi) \cdot (D\Phi)^{-T}\nabla(v \circ \Phi)|J\Phi| = \int_\Omega (f \circ \Phi)(v \circ \Phi)|J\Phi| \quad \forall v \circ \Phi \in H_0^1(\Omega) \tag{3.1}$$

where $D\Phi$ is the Frechet derivative of order one which agrees with the Jacobian matrix. It is clear that through Φ we recover the exact interface by using curvilinear elements in the domain of the chart.

The Galerkin discretization corresponds to the mapped linear Lagrangian finite elements and gives the solution $\hat{u}_h = u_h \circ \Phi$. Next step is to estimate the errors between the original solution u and $\hat{u}_h$ in the energy norm. Taking into account the perturbed form of the problem and consistency of the bilinear forms, we are lead to the following error estimate involving first Strang's lemma, which in our case reads

$$\|u-\hat{u}_h\|_{H^1(\Omega)} \leq C\Big\{ \inf_{\hat{v}_h\in\hat{V}_h}\Big(\|u-\hat{v}_h\|_{H^1(\Omega)} + \sup_{\hat{w}_h\in\hat{V}_h} \frac{\left|\int_\Omega f\hat{w}_h dx - \int_\Omega (f\circ\Phi)|J\Phi|\hat{w}_h dx\right|}{\|\hat{w}_h\|_{H^1(\Omega)}}$$

$$+ \sup_{\hat{w}_h\in\hat{V}_h} \frac{\left|\int_\Omega c(x)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx - \int_\Omega (c(x)\circ\Phi)(D\Phi)^{-T}\nabla(\hat{v}_h)\cdot(D\Phi)^{-T}\nabla(\hat{w}_h)|J\Phi|dx\right|}{\|\hat{w}_h\|_{H^1(\Omega)}} \Big)\Big\} \tag{3.2}$$

where $\hat{S}_1^0 = \{\hat{v}_h = v_h\circ\Phi | v_h \in S_1^0\} = \hat{V}_h$.

3.1 The error components

In this subsection we give concrete estimates for the various error components. We begin by introducing a lemma from [1] that establishes a connection between the norms of the discrete and discrete perturbed solutions.

Remark 3.1. We remark that localization for the data, bilinear form and discrete solution - perturbed discrete solution error, refers to an element in the domain of the chart, since in the other elements, Φ is just the identity. Localization takes place to elements K of the initial triangulation. The notation H^i and $W^{k,m}$ stands for the standard Sobolev spaces. Finally, whenever we use Sobolev norms that refer to whole Ω, i.e. $\| \|_{H^i(\Omega)}$, we refer to the broken Sobolev norms.

Lemma 3.2. *The norms $\|u_h\|_{H^m(K)}$, $\|u_h\circ(\Phi)^{-1}\|_{H^m(\Psi_K(K))}$ are uniformly, with respect to h, equivalent for $m\leq 2$*

Proof. Follows from Proposition 2.2 and the classical inequality of Ciarlet and Raviart [10]

$$|u_h\circ(\Phi)^{-1}|_{H^m(\Psi_K(K))} \leq C \sup_{M\in\Psi_K(K)} |J\Phi_M| \sum_{i=1}^m |u_h|^2_{H^i(\Psi_K(K))}$$
$$\times \sum_{j\in I(i,m)} |||D((\Phi)^{-1})|||^{2j_i}_{\Psi_K(K)} \cdots |||D^m((\Phi)^{-1})|||^{2j_m}_{\Psi_K(K)}$$

where $|||D^m((\Phi)^{-1})|||$ is the sup-norm of the Frechet derivative of m-th order
□

The next lemmas give error estimates for the interpolation error the data approximation error and the bilinear form approximation error respectively. Throughout these lemmas we keep the notation C for all constants, in order to avoid terms of the form C', C'' etc, which are only important, if the best constant is requested, something that is irrelevant to our work.

Lemma 3.3. *The following estimate holds:*

$$\|u - \hat{u}_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)} \tag{3.3}$$

where $H^i(\Omega)$ is understood as the broken Sobolev norm

Proof. First we note that we do not have regularity problems due to the use of the broken Sobolev norm. Hence, the H^2-norm estimate is meaningful. The classical interpolation error estimates then are applied and the estimate follows from Proposition 2.2 and lemma 3.1 $\square$

Lemma 3.4. *Assume that $f \in L^\infty$. Then the following estimate holds for the data approximation*

$$\sup_{\hat{w}_h \in \hat{V}_h} \frac{\left| \int_\Omega f \hat{w}_h dx - \int_\Omega (f \circ \Phi)|J\Phi| \hat{w}_h dx \right|}{\|\hat{w}_h\|_{H^1(\Omega)}} \leq Ch\|f\|_{L^\infty(\Omega)} \tag{3.4}$$

Proof. Caucy-Schwarz implies

$$\sup_{\hat{w}_h \in \hat{V}_h} \frac{\left| \int_\Omega f \hat{w}_h dx - \int_\Omega (f \circ \Phi)|J\Phi| \hat{w}_h dx \right|}{\|\hat{w}_h\|_{H^1(\Omega)}} \leq \frac{\|\hat{w}_h\|_{L^2(\Omega)} \|f - (f \circ \Phi)|J\Phi|\|_{L^2(\Omega)}}{\|\hat{w}_h\|_{H^1(\Omega)}}$$

The trivial H^1 embedding into L^2 allows us to remove the $\hat{w}_h$ norms and hence, what is left to estimate is the term

$$\|f - (f \circ \Phi)|J\Phi|\|_{L^2(\Omega)}$$

We localize into an element K. Then we obtain:

$$\begin{aligned} \|f - (f \circ \Phi)|J\Phi|\|_{L^2(K)} &= \|f - (f \circ \Phi)|J\Phi| + f|J\Phi| - f|J\Phi|\|_{L^2(K)} \\ &\leq \|f - f|J\Phi|\|_{L^2(K)} + \||J\Phi|(f - (f \circ \Phi))\|_{L^2(K)} \end{aligned} \tag{3.5}$$

For the first term of (3.5) , since both the Jacobian and f are bounded, we have

$$\|f - f|J\Phi|\|_{L^2(K)} \leq \|1 - |J\Phi|\|_{L^\infty(K)} \|f\|_{L^\infty(K)}$$

Now Proposition 2.2 implies

$$\|1 - |J\Phi|\|_{L^\infty(K)} \|f\|_{L^\infty(K)} \leq Ch\|f\|_{L^\infty(K)} \leq Ch\|f\|_{L^\infty(\Omega)}$$

For the second term of (3.5), we first observe that by construction of our triangulation if the element K lies in the domain of the chart then it is going

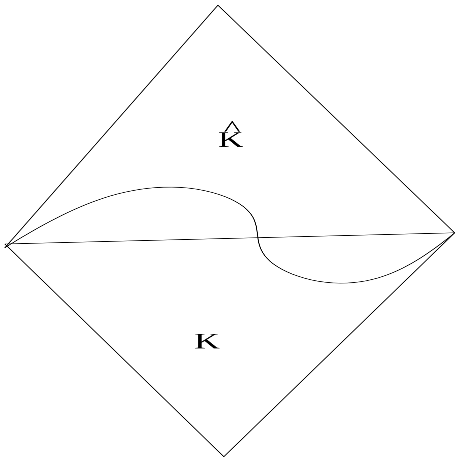

Fig. 6. K and $\hat{K}$

to by adjusted to only one element that lies in the other side of the partition (see for example figure 6), let it be $\hat{K}$. Under this notation we have

$$\||J\Phi|(f-(f\circ\Phi))\|_{L^2(K)} \leq \||J\Phi|(f-(f\circ\Phi))\|_{L^2(K\cup\hat{K})}$$

Now this step has been done in order to take into account the case where $f(\Phi(x))$ does not lie in K but in $\hat{K}$, i.e $K \subset \Phi(K)$. Since, now $f \in L^\infty(\Omega)$ we can bound the last part of the inequality as follows

$$\||J\Phi|(f-(f\circ\Phi))\|_{L^2(K\cup\hat{K})} \leq \||J\Phi|(\sup_{x\in K\cup\hat{K}}\{f\} - \inf_{x\in K\cup\hat{K}}\{f\})\|_{L^2(K\cup\hat{K})}$$

which implies

$$\leq \||J\Phi|\|f\|_{L^\infty(K\cup\hat{K})}\|_{L^2(K\cup\hat{K})} \leq \|J\Phi\|_{L^\infty(K\cup\hat{K})}\|f\|_{L^\infty(K\cup\hat{K})}(|K\cup\hat{K}|)^{1/2}$$

where $|K\cup\hat{K}|$ stands for the area of $K\cup\hat{K}$. Now the area of $K\cup\hat{K}$ is of order h^2 and together with Proposition 2 we obtain

$$\begin{aligned} &\leq C(1+h)h\|f\|_{L^\infty(K\cup\hat{K})} \\ &\quad \leq C(1+h)h\|f\|_{L^\infty(\Omega)} \end{aligned}$$

Now we can add the estimates of the two expressions of (3.5) and finally we obtain after summing up for all elements

$$\|f-(f\circ\Phi)|J\Phi|\|_{L^2(\Omega)} \leq Ch\|f\|_{L^\infty(\Omega)}$$

since the rate of convergence is dominated by the smaller power of h. The lemma follows. □

Lemma 3.5. *For the approximation of the bilinear form the following estimate* *holds*

$$\sup_{\hat{w}_h \in \hat{V}_h} \frac{\left| \int_\Omega c(x)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx - \int_\Omega (c(x)\circ\Phi)(D\Phi)^{-T}\nabla(\hat{v}_h)\cdot(D\Phi)^{-T}\nabla(\hat{w}_h)|J\Phi|dx \right|}{\|\hat{w}_h\|_{H^1(\Omega)}} \leq Ch\|u_h\|_{H^1(\Omega)} \tag{3.6}$$

Proof. We have to estimate the expression

$$|\int_\Omega c(x)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx - \int_\Omega (c(x)\circ\Phi)(D\Phi)^{-T}\nabla(\hat{v}_h)\cdot(D\Phi)^{-T}\nabla(\hat{w}_h)|J\Phi|dx| \tag{3.7}$$

We localize now to an element K, and we have

$$\begin{aligned} &|\int_K c(x)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx - \int_K (c(x)\circ\Phi)(D\Phi)^{-T}\nabla(\hat{v}_h)\cdot(D\Phi)^{-T}\nabla(\hat{w}_h)|J\Phi|dx| \\ &= |\int_K (c(x) - (c(x)\circ\Phi)(D\Phi)^{-T}(D\Phi)^{-1}|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \\ = \|\int_K (c(x) - c\circ\Phi(x)(D\Phi)^{-T}(D\Phi)^{-1}|J\Phi| &+ c\circ\Phi(x)|J\Phi| - c\circ\Phi(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \end{aligned}$$

Triangle inequality implies

$$\begin{aligned} &\leq |\int_K (c(x) - c\circ\Phi(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \\ +|\int_K (c\circ\Phi(x)|J\Phi| &- c\circ\Phi(x)(D\Phi)^{-T}(D\Phi)^{-1}|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \end{aligned}$$

Adding and substracting $c(x)|J\Phi|$ gives

$$\begin{aligned} = |\int_K (c(x) - c\circ\Phi(x)|J\Phi| + c(x)|J\Phi| - c(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \\ +|\int_K c\circ\Phi(x)|J\Phi|(I - (D\Phi)^{-T}(D\Phi)^{-1})\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \end{aligned}$$

Again Triangle inequality gives

$$
\begin{aligned}
&\leq |\int_K (c(x) - c(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| + |\int_K (c(x)|J\Phi| - c\circ\Phi(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \\
&\qquad + |\int_K c\circ\Phi(x)|J\Phi|((D\Phi)^{-T}(D\Phi)^T - (D\Phi)^{-T}(D\Phi)^{-1})\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)| \\
&= |\int_K (c(x) - c(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| + |\int_K (c(x)|J\Phi| - c\circ\Phi(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \\
&\qquad + |\int_K c\circ\Phi(x)|J\Phi|(D\Phi)^{-T}((D\Phi)^T - (D\Phi)^{-1})\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \\
&= |\int_K (c(x) - c(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| + |\int_K (c(x)|J\Phi| - c\circ\Phi(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \\
&\qquad + |\int_K c\circ\Phi(x)|J\Phi|(D\Phi)^{-T}((D\Phi)^T - (D\Phi)^{-1} + I - I)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx|
\end{aligned}
$$

One more time triangle inequality gives

$$
\begin{aligned}
&\leq |\int_K (c(x) - c(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| + |\int_K (c(x)|J\Phi| - c\circ\Phi(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \\
&\qquad + |\int_K c\circ\Phi(x)|J\Phi|(D\Phi)^{-T}((D\Phi)^T - I)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| \\
&\qquad + |\int_K c\circ\Phi(x)|J\Phi|(D\Phi)^{-T}(I - (D\Phi)^{-1})\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx|
\end{aligned} \tag{3.8}
$$

For the first term of (3.8) we have

$$
|\int_K (c(x) - c(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| = |\int_K c(x)(1 - |J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx|
$$

Using Proposition 2.2 and the fact that the conductivity tensor is bounded gives

$$
\leq Ch\int_K |\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)|dx \tag{3.9}
$$

$$
\tag{3.10}
$$

For the second term of (3.8) we have

$$
|\int_K (c(x)|J\Phi| - c\circ\Phi(x)|J\Phi|)\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx| = |\int_K |J\Phi|(c(x) - c\circ\Phi(x))\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx|
$$

Proposition 2.2 gives

$$
\leq C|\int_K (c(x) - c\circ\Phi(x))\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx|
$$

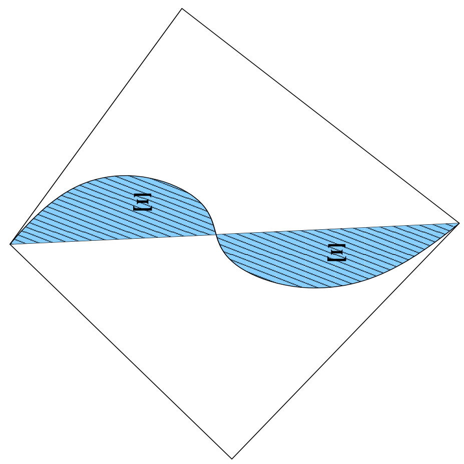

Fig. 7. The area Ξ

Now, we extend our localization to $K \cup \hat{K}$ like before and we have

$$\leq C|\int_{K\cup\hat{K}}(c(x) - c\circ\Phi(x))\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx|$$

Now we notice that $c(x) - c\circ\Phi(x) \neq 0$ if and only if we are in the area Ξ of figure 7

Therefore we have

$$= C|\int_{\Xi}(c(x) - c\circ\Phi(x))\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx|$$
$$= C|\frac{|\Xi|}{|K\cup\hat{K}|}\int_{K\cup\hat{K}}(c(x) - c\circ\Phi(x))\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)dx|$$

Since the conductivity tensor is bounded and the area of Ξ and $K \cup \hat{K}$ is of order h^3 and h^2 respectively, we have

$$\leq Ch\int_{K\cup\hat{K}}|\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)|dx \tag{3.11}$$

The third and fourth term of (3.8) can be both estimated using Proposition 2.2 and the fact that the conductivity tensor is bounded by

$$\leq Ch\int_{K\cup\hat{K}}|\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)|dx \tag{3.12}$$

(3.9),(3.11) and (3.12) imply that (3.8) can be estimated by

$$\leq Ch\int_{K\cup\hat{K}}|\nabla(\hat{v}_h)\cdot\nabla(\hat{w}_h)|dx$$

Cauchy-Schwarz implies

$$\leq Ch\|\nabla(\hat{v}_h)\|_{L^2(K\cup\hat{K})}\|\nabla(\hat{w}_h)\|_{L^2(K\cup\hat{K})}$$

which can be trivially bounded by

$$\leq Ch\|\nabla(\hat{v}_h)\|_{L^2(\Omega)}\|\nabla(\hat{w}_h)\|_{L^2(\Omega)}$$
$$\leq Ch\|\nabla(\hat{v}_h)\|_{L^2(\Omega)}\|\nabla(\hat{w}_h)\|_{H^1(\Omega)}$$

Summing up over all elements we finnaly have an estimate for (3.7)

$$\leq Ch\|\nabla(\hat{v}_h)\|_{L^2(\Omega)}\|\nabla(\hat{w}_h)\|_{H^1(\Omega)} \tag{3.13}$$

Now (3.13) gives an estimate for the left part of (3.6)

$$\leq Ch\|\nabla(\hat{v}_h)\|_{L^2(\Omega)}$$

We now note that $\hat{v}_h$ in the last expression comes from the infinmum of Strang's lemma. Therefore, using $\hat{u}_h$ instead can only increase or leave unchanged the estimate. Finally, due to lemma 3.2 we can replace the L^2-norm of $\hat{u}_h$ with the L^2-norm of u_h thus we finally have the estimate

$$\leq Ch\|\nabla(u_h)\|_{L^2(\Omega)}$$

The lemma follows □

We now collect all the results of this section to formulate the final estimate for the energy norm error $\|u - \hat{u}_h\|_{H^1(\Omega)}$

Proposition 3.6. *Under the assumptions of the previous sections the following estimate holds*

$$\|u - \hat{u}_h\|_{H^1(\Omega)} \leq Ch(\|f\|_{L^\infty(\Omega)} + \|u\|_{H^2(\Omega)}) \tag{3.14}$$

Proof. From lemmas 3.1-3.4 and Note .2 we have the following estimate

$$\|u - \hat{u}_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)} + Ch\|u_h\|_{H^1(\Omega)} + Ch\|f\|_{L^\infty(\Omega)} \tag{3.15}$$

Since, we use linear Lagrangian finite elements we can bound the $\|u_h\|_{H^1(\Omega)}$ norm from $\|u\|_{H^1(\Omega)}$, therefore, from $\|u\|_{H^2(\Omega)}$, in the broken sense as we have mentioned. Proposition follows. □

3.2 The error between the two discrete solutions

In this subsection we estimate the error between u_h and $\hat{u}_h$ in the energy norm. The next Proposition gives the required estimate.

Proposition 3.7. *Under the notation and assumptions of the previous sections the following estimate holds*

$$\|u_h - \hat{u}_h\|_{H^1(\Omega)} \le Ch(\|u_h\|_{H^1(\Omega)} + \|u_h\|_{W^{1,\infty}(\Omega)}) \tag{3.16}$$

Proof. We will treat the L^2 norm and the H^1 seminorm seperately. For the L^2 norm we have after localization to an element K

$$\|u_h - \hat{u}_h\|_{L^2(K)} \le \|u_h - \hat{u}_h\|_{L^2(K\cup\hat{K})}$$

where $\hat{K}$ denotes the adjusted element of K in the interface, as in lemma 3.3. Now the mean value theorem gives

$$= \left(\int_{K\cup\hat{K}} \left|\int_0^1 \nabla(u_h(x+\tau(\Phi(x)-x)))(\Phi(x)-x)d\tau\right|^2 dx\right)^{1/2}$$

Since the gradient is piecewise constant and $(\Phi(x)-x)$ is of order h, we have the upper bound

$$\le Ch\|\nabla(u_h)\|_{L^2(K\cup\hat{K})} \le Ch\|\nabla(u_h)\|_{L^2(\Omega)}$$

This can be trivially bounded from

$$\le Ch\|u_h\|_{H^1(\Omega)}$$

Summing up over all elements, gives the estimate for the L^2 norm of the error

$$\le Ch\|u_h\|_{H^1(\Omega)} \tag{3.17}$$

For the H^1 seminorm we have after localization to $K \cup \hat{K}$

$$\begin{aligned} &\|\nabla((u_h - \hat{u}_h)(x))\|_{L^2(K\cup\hat{K})} = \|\nabla(u_h(x)) - D\Phi^T\nabla(u_h(\Phi(x)))\|_{L^2(K\cup\hat{K})} \\ &= \|\nabla(u_h(x)) - D\Phi^T\nabla(u_h(\Phi(x))) + D\Phi^T\nabla(u_h(x)) - D\Phi^T\nabla(u_h(x))\|_{L^2(K\cup\hat{K})} \\ &\le \|(I - D\Phi^T)\nabla(u_h(x))\|_{L^2(K\cup\hat{K})} + \|D\Phi^T(\nabla(u_h(x)) - \nabla(u_h(\Phi(x)))\|_{L^2(K\cup\hat{K})} \end{aligned} \tag{3.18}$$

The first term of (3.18) can be estimated using Proposition 2.2 from

$$\le Ch\|\nabla(u_h(x))\|_{L^2(K\cup\hat{K})}$$

Summing up over all elements we obtain the bound

$$\le Ch\|u_h\|_{H^1(\Omega)} \tag{3.19}$$

For the second term of (3.18), we first observe that it is non zero if and only if x and $\Phi(x)$ belong to different elements i.e. $\Phi(x) \in \Xi$, where Ξ is the shaded area of figure 7.

This implies that we can bound the second term by

$$\leq \left(\int_{\Xi} |\nabla u_h(x) - \nabla u_h(\Phi(x))|^2 \right)^{1/2}$$

Now the area of Ξ is of order h^3 while the area of $K \cup \hat{K}$ is of order h^2. This gives

$$= \left(\frac{E(\Xi)}{E(K \cup \hat{K})} \int_{K\cup\hat{K}} |\nabla u_h(x) - \nabla u_h(\Phi(x))|^2 dx \right)^{1/2} \leq \left(h \int_{K\cup\hat{K}} |\nabla u_h(x) - \nabla u_h(\Phi(x))|^2 dx \right)^{1/2}$$

Since the gradient is bounded and the area of $K \cup \hat{K}$ is of order h^2 we can estimate the last expression by

$$\leq \left(Ch^3 \|\nabla u_h(x)\|^2_{L^\infty(K\cup\hat{K})} \right)^{1/2}$$

which can be trivially bounded by

$$\leq Ch^{3/2} \|u_h\|_{W^{1,\infty}(K\cup\hat{K})}$$

We finally bound, after summing up over all elements, the last term by

$$\leq Ch^{3/2} \|u_h\|_{W^{1,\infty}(\Omega)} \tag{3.20}$$

Therefore the H^1 part of the error is bounded due to (3.19),(3.20) by

$$\leq Ch(\|u_h\|_{H^1(\Omega)} + \|u_h\|_{W^{1,\infty}(\Omega)}) \tag{3.21}$$

The lemma follows from (3.21) and (3.17) □

3.3 The final error estimate

We can now formulate the requested error estimate. This is described in the following theorem

Theorem 3.8. *Under the assumptions and notation of the previous sections the following estimate holds*

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch(\|f\|_{L^\infty(\Omega)} + \|u\|_{H^2(\Omega)} + \|u\|_{W^{1,\infty}(\Omega)}) \tag{3.22}$$

Proof. Adding and subtracting the factor $\hat{u}_h$ in the error norm and triangle inequality reveal the already familiar error estimates. In the spirit of Proposition 3.5 we replace the H^1 norm of u_h, with the H^2 norm of u. The Theorem follows. □

4 Conclusion

In this paper we discussed the interface problem for elliptic boundary value problems in 2-d. We proposed a new conforming method based on mapping techniques and we presented the error analysis. Optimal rate of convergence was recovered for linear Lagrangian finite elements. Although, the method was briefly discussed and developed for linear elements it can be generalized to elements of arbitrary order. Generalization of our results to the 3-d case is possible, due to the beautiful construction of the Φ mapping, but this will be discussed in detail in the second part of the paper which will be focused on the 3-d case.

The most important question that is left open comes from the necessary restriction that we required for the data function f, namely $f \in L^\infty(\Omega)$. At first glance it is not clear whether this condition can be weakened, therefore, it would be interesting to come up with a proof that claims the inverse and maintains the same rate of convergence. Also, the $W^{1,\infty}$-norm is not satisfying however, the usual error analysis lead to lower rate of convergence ($h^{1/2}$), when the usual H^2 norm is used instead and moreover, this ratae seems to be sharp.

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5 Appendix A: The construction of Φ

The content of the Appendix is borrowed from [1], however, only the details that are of a interest to our approach are presented. Hence, in some points we give the general formulas as originally in [1] and in some points we restrict ourselves to specific examples.

The first part discusses the construction of isoparametric finite elements from a reference element in a domain with an interface.

The elements in the physical plane

For the simplicial Lagrangian element of reference of order k we fix the notation

- $\hat{K}$ for the triangle of reference
- $\hat{\Sigma} = \{\hat{a}_k^i \in \hat{K} | i = 1, .., 3\}$ the set of nodal points
- $\hat{P_k}$= set of polynomial of two variables of total degree k

Let F_k^m denote a regular, invertible mapping over $\hat{K}$. Then the elements of in the physical domain are obtained by $\hat{K}$ through the formula:

$$F_k^m(\hat{K}) = K^m \tag{5.1}$$

we also fix the notation:

- $\Sigma^m = \{F_k^m(\hat{a}_k^i) | i = 1, ..3\}$
- $P^m = \{\hat{p_k} \circ (F_k^m)^{-1} | \hat{p_k} \in P_k\}$

for the nodal points and polynomials correspondingly.

Note 2. The mapping F_k^m is generated by interpolation over the reference triangle in the calculation domain under the formula

$$F_K^m = \sum_1^3 F_K^m(a_m^i) b_m^i \tag{5.2}$$

where a_m^i, b_m^i are the nodal values and the basis functions of the reference element of the calculation domain.

The elements in the domain of the chart

For the construction of the elements in the domain of the chart we need to go through 1-dimensional elements ((n-1)-dimensional in the general case of $\mathbb{R}^n$). Again the elements are going to be obtained by the reference element of the calculation domain through a mapping that is connected to an interpolation operator, and furthermore, the obtained elements are going to be curvilinear.

We fix the notation:

- $\hat{k}$ for the 1-d reference element of order m of the calculation domain
- k for the curvilinear elements

We also denote

$$G^l_{\hat{k}} : \hat{k} \to k^l \tag{5.3}$$

the mapping that gives the curvilinear elements of order l generated by interpolation similar to F^m_k. Moreover, Λ^l_m will denote the interpolation operator over the element k^l which is normally satisfies the formula

$$\Lambda^l_m(\psi) \circ G^l_{\hat{k}} = \hat{\Lambda}_m(\psi \circ G^l_{\hat{k}}), \ \ \psi \in C^0(k^i) \tag{5.4}$$

where $\hat{\Lambda}_m$ denotes the interpolation over the reference element $\hat{k}$

The target is to obtain the elements of the chart through interpolation of the map ϕ. Let now λ_i, μ_j denote the barycentric coordinates of the elements $\hat{K}, \hat{k}$ respectively.

Let now K^m belong in the domain of the chart. In the 2-d case that we study the analysis is simplified since there is only one way that a triangle K can intersect Γ. Thus if we denote by $\gamma^1 K^m$ the 1-face that K^m intersects γ we will require that this element is connected with one element k^l by the formula

$$F^m_{K|\gamma^1\hat{K}} = \Lambda^l_m(\phi) \circ G^l_{\hat{k}} \circ \mathcal{F}|\gamma^1\hat{K} \tag{5.5}$$

where $\mathcal{F}$ is the natural isomorphism between $\gamma^1\hat{K}$ and $\hat{k}$ defined by:

$$N = \mathcal{F}(M) \tag{5.6}$$

$$\mu_i(N) = \lambda_i(M), \forall M \in \gamma^1\hat{K} \tag{5.7}$$

These definitions may seem abstract at first however, they rise in a natural way if one tries to construct the problem geometrically in 2 dimensions.

5.1 An algorithm for the construction of F^m

The algorithm we are going to present here is the 2-d restriction of a more general algorithm that has proven to work for arbitrary dimensions. It has been established by M. Lenoir in [1]. One important remark about the algorithm, is that it leads to finite elements that satisfy the conditions of the Ciarlet-Raviart theorem [10] for optimal finite elements.

In the 2-d case, as already mentioned, the analysis is simple due to the unique way that a triangle can intersect Γ. We first assume that a 1-face of of an element K^1 lays along the interface. We need to define a mapping

$$Z^p : \hat{K} \to \gamma^1 \hat{K} \tag{5.8}$$

which maps the barycentric coordinates of $M \in \hat{K}$, $(\lambda_1, \lambda_2, \lambda_3)$ to the point $Z^p(M) \in \gamma^1 \hat{K}$ with barycentric coordinates $(\frac{\lambda_1}{1-\lambda_3}, \frac{\lambda_2}{1-\lambda_3}, 0)$.

Now we are ready to define the mapping F_k^m. Since we are restricted in 2-dimensions it is safe to assume that we have an affine initial triangulation of Ω, no change of coordinates in the domain of the chart is required.

Let $F_K^t : \hat{K} \to K^t$ be a regular affine family of triangulations of Ω. Then for every element K^t we put naturally:

(1) $F_K^m = F_K^t$, if no face of K^t belongs to the interface
(2) if an 1-face of K^t belongs to the interface then by the assumption that

$$\gamma^1 K^t = F_K^t(\gamma^1 \hat{K}) \tag{5.9}$$

we define

$$F_K^m = (1-\lambda_3)^m (\Lambda_m^l(\phi) - \Lambda_{m-1}^l(\phi)) \circ G_{\hat{k}}^l \circ \mathcal{F} \circ Z^1 + F_K^{m-1} \quad \forall m \geq 1 \tag{5.10}$$

For convenient reasons we define $F_K^0 = 0$. This will simplify the error analysis of linear Lagrangian finite elements in the next section.

The next Lemma and Theorem give a full description of the mapping F_K^m. The proofs can be found in [1]

Lemma 5.1. *If $\gamma^1 K^t = \gamma^1 F_K^t(\hat{K})$ belongs to the approximate boundary of the interface of Ω then the following are true*

(1) $F^m_{K|\gamma^1\hat{K}} = \Lambda_m^l(\phi) \circ G_{\hat{k}}^l \circ \mathcal{F}_{|\gamma^1\hat{K}}$
(2) $F_K^m = F_K^t$ *along the faces of $\hat{K}$ which do not have more than one common point with $\gamma^1\hat{K}$*

The first part of the lemma proves that in the way F_K^m was defined the condition (5.5) is satisfied. The second part proves the compatibility along the common face of two elements such as the first has a face in the approximate interface and the second not.

Theorem 5.2. *The mapping F_K^m satisfies the conditions of the Ciarlet-Raviart theorem [10]*

6 Mapping Ω to Ω

From here now we will only consider isoparametric finite elements of order one i.e $m = k = 1$ in the mapping F_k^m. This restriction also applies to the 1-d mappings, introduced before.

The mapping Φ will be defined through its restriction to every element. What is also worth mentioning is that Φ is a local mapping, in a sense that the its values in an element depend only on this element.

Definition 6.1. Let Ψ_K denote the restriction of Φ in the element K. Then:

(1) If not face of K belongs to the interface approximation, we set $\Psi_K = I$
(2) Else if one 1-face of K, say $\gamma^1 K$ belongs to the interface approximation, we set

$$(\Psi_K - I) \circ F_K^1 = (1 - \lambda_3)^2(\phi - \Lambda_1^1(\phi)) \circ G_{\hat{k}}^1 \circ \mathcal{F} \circ Z^p \tag{6.1}$$

where F_K^1 are the mappings we introduced in the previous sections

Figure 7 demonstrates the operation of all the mappings that have been used in the construction of Φ

The next lemma proves the compatibility of Φ along the common face of two adjusted elements.

Lemma 6.2. *If $\gamma^1 K^1$ belongs to the interface approximation the following hold*

(1) $\Psi_K \circ \Lambda_1^1(\phi) \circ G_{\hat{k}}^1 \circ \mathcal{F}_{|\gamma^1 \hat{K}} = \phi \circ G_{\hat{k}}^1 \circ \mathcal{F}_{|\gamma^1 \hat{K}}$
(2) $\Psi_K = I$ *along the faces of K^1 which do not have more than one common point with $\gamma^1 K^1$*

An immediate corollary of this lemma is the following

Corollary 6.3. Φ*, as defined locally through the mappings Ψ_K maps Ω into Ω.*

Remark 6.4. The advantage of defining Φ as in formula 3.1 is based on the fact that is connected with mapping F^1, in general F_k^m which it has been proved to have good properties and that it is a polynomial up to degree 1, in general up to degree m.

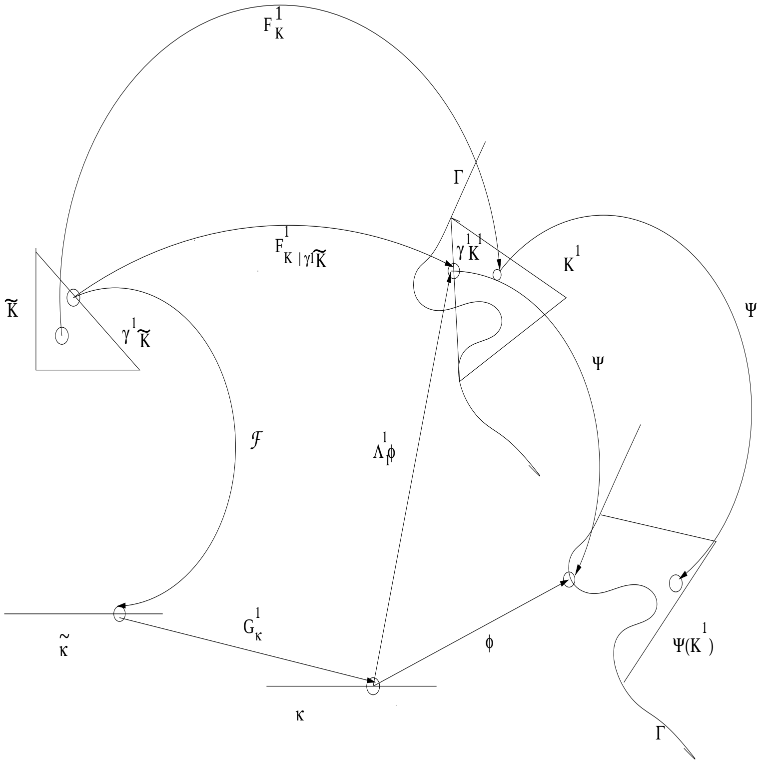

Fig. 8. Operation of all mappings

7 Appendix B : Strang's lemma

Let V = Hilbert space and $V_0 \subset V$ closed subspace in which we have the linear variational problem

$$u \in V_0 : a(u,v) = f(v) \forall v \in V_0 \tag{7.1}$$

under the assumptions

- $a \in L(V \times V, \mathbb{K})$ continuous sesquilinear form
- $f \in V'$

We do Galerkin discretization with trial space $V_N \subset V$ which implies that generally $V_N \not\subset V_0$

The above imply that we get an approximate sesquilinear form a_N and an approximate right hand side f_N. Hence, we have the following discretized perturbed problem

$$uN \in V_N : a_N(u_N, v_N) = f_N(v_N) \ \ \forall v_N \in V_N \tag{7.2}$$

Error analysis leads to first Strang's lemma

Theorem 7.1. ***Strang's lemma***
Let $u \in V_0$ and $u_N \in V_N$ denote the solutions of the initial variational problem and the perturbed one respectively. Furthermore we assume V_N-ellipticity for the perturbed sesquilinear form with ellipticity constant $\tilde{\gamma}$. Then we have the a-priori error estimate

$$\|u - u_N\|_V \leq \inf_{v_N \in V_N} \left((1 + \frac{\|a\|}{\tilde{\gamma}})\|u - v_N\|_V + \frac{1}{\tilde{\gamma}} \sup_{w_N \in V_N} \frac{|a(v_N, w_N) - a_N(v_N, w_N)|}{\|w_N\|_V} \right) + \frac{1}{\tilde{\gamma}} \sup_{w_N \in V_N} \frac{|a(u, w_N) - f_N(w_N)|}{\|w_N\|_V} \tag{7.3}$$