Master Thesis Project

(Mathematics, Computational Science & Engineering)

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Vector Wave Propagation along a Sphere

Prerequisites. Knowledge about the Boundary Element method for the solution of the Helmholtz equation [11].

Problem description. Starting from Maxwell's equation, the electric wave equation $\mathbf{curl}\,\mathbf{curl}\boldsymbol{E} - k^2\boldsymbol{E} = 0$ can be easily derived. It describes propagation of electromagnetic waves in terms of the electric field $\boldsymbol{E}$ [1]. Herein, $k \in \mathbb{R}^+$ is the wave number.

This setting can be generalized by replacing $\boldsymbol{E}$ by a differential 1-form $\boldsymbol{\omega}$, subject to the Maxwell-type equation $(\delta\mathrm{d} - k^2)\boldsymbol{\omega} = 0$ [7]. Herein, d denotes the exterior derivative, and δ the co-derivative, its formal L^2 adjoint [2,8]. The latter equation seamlessly extends to dimensions n other than 3. Moreover, it can be posed in Riemannian manifolds, while the original double **curl** equation relies on flat Euclidean space.

In this Master Thesis project, as a model problem, wave propagation for differential 1-forms on a sphere $\mathcal{S}$ with radius R shall be considered, i.e., $n = 2$. A point source radiates an incident wave $\boldsymbol{\omega}_\mathrm{i}$, which impinges on a perfect scatterer $\Omega \subset \mathcal{S}$, an "island" on the sphere. Ω is required to be connected and simply connected, with sufficiently smooth boundary $\Gamma = \partial\Omega$. Let $\Omega' = \mathcal{S}\backslash\overline{\Omega}$ denote the complement of Ω in $\mathcal{S}$. This situation gives rise to a Dirichlet problem for the scattered field $\boldsymbol{\omega} \in H\Lambda^1(\delta\mathrm{d}, \Omega')$ [7]

$$\left.\begin{aligned} (\delta\mathrm{d} - k^2)\boldsymbol{\omega} &= 0, \\ \mathbf{t}\boldsymbol{\omega} &= -\mathbf{t}\boldsymbol{\omega}_\mathrm{i}, \end{aligned}\right\} \tag{1}$$

where $\mathbf{t} : H\Lambda^p(\delta\mathrm{d}, \Omega') \to H_\perp^{-1/2}\Lambda^p(\mathrm{d}, \Gamma)$ is the tangential trace.

The boundary value problem can be recast into a boundary integral equation, which lives on the one-dimensional boundary Γ. To that end, a fundamental solution of a Helmholtz-type equation is required. Denote x the observation point and y the source point. We exclude the case where $x, y \in \mathcal{S}$ are antipodal points. There is a unique minimal geodesic that connects x to y. Denote the geodesic distance by $s(x, y)$, and the geodesic propagator P_x^y. The geodesic propagator parallel transports a given tangent vector from y to x along the geodesic. Note that a canonical coordinate representation of P_x^y is achieved if Fermi normal coordinates are used, since they are adapted to the geodesic [10].

At this point it is convenient to introduce double forms [3, pp. 30-33]. A double p-form is defined for $p > 0$ by its action on a pair of p-tupels of tangent vectors anchored in x and y, respectively. Double 0-forms are simply complex two-point functions. We are mainly concerned with the cases $p = 0, 1$. We denote $I_p(x, y)$ the identity double p-form, which is defined for $p = 1$ by $I_1(x, y)[\mathbf{t}_x, \mathbf{t}_y] = g(\mathbf{t}_x, P_x^y \mathbf{t}_y)$, where $\mathbf{t}_x$, $\mathbf{t}_y$ are tangent vectors, and $g(\cdot, \cdot)$ is the metric tensor. Moreover, $I_0(x, y) = 1$. The Helmholtz-Green kernel double p-form $G_p(x, y)$ is defined by $(\Delta_y - k^2)G_p(x, y) = \delta(x, y)I_p(x, y)$, where $\Delta = \mathrm{d} \circ \delta + \delta \circ \mathrm{d}$ is the Hodge Laplacian, and $\delta(x, y)$ the Dirac delta distribution. The fundamental solution can be written in the form $G_p(x, y) = w_p\big(s(x, y)\big)I_p(x, y)$, where each $w_p(s)$ is related to a solution of the hypergeometric equation [4, 6].

We are now in a position to define the single layer potential $\Psi_{\mathrm{SL},p} : H_{\|}^{-1/2}\Lambda^p(\delta, \Gamma) \to H\Lambda^p(\delta\mathrm{d}, \Omega')$ [7, (21a)]:

$$\boldsymbol{\gamma} \mapsto \Psi_{\mathrm{SL},p}(\boldsymbol{\gamma})(x) = \int_\Gamma \langle G_p(x, y), \boldsymbol{\gamma}(y)\rangle \,\mathrm{d}\Gamma(y), \quad x \notin \Gamma,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product for p-covectors. The Maxwell single layer and double layer potentials ensue [1, (27),(28)], [7, (72),(21b)], ($n = 2$, $p = 1$)

$$\begin{aligned}
\widetilde{\Psi}_{\mathrm{SL},p} &: H_{\|}^{-1/2}\Lambda^p(\delta, \Gamma) \to H\Lambda^p(\delta\mathrm{d}, \Omega') : \boldsymbol{\gamma} \mapsto \Psi_{\mathrm{SL},p}(\boldsymbol{\gamma}) - \frac{1}{k^2}\mathrm{d}\Psi_{\mathrm{SL},p-1}(\delta\boldsymbol{\gamma}),\\
\Psi_{\mathrm{DL},p} &: H_{\perp}^{-1/2}\Lambda^p(\mathrm{d}, \Gamma) \to H\Lambda^p(\delta\mathrm{d}, \Omega') : \boldsymbol{\beta} \mapsto - * \mathrm{d}\Psi_{\mathrm{SL},n-1-p}(\hat{*}^{-1}\boldsymbol{\beta}),
\end{aligned}$$

where $\hat{*}$ and $*$ are the Hodge operators related to Γ and Ω', respectively.

By taking the traces we obtain the Maxwell single layer and double layer operators,

$$\begin{aligned}
\widetilde{V} &= \mathbf{t} \circ \widetilde{\Psi}_{\mathrm{SL}} &&: H_{\|}^{-1/2}\Lambda^p(\delta, \Gamma) \to H_{\perp}^{-1/2}\Lambda^p(\mathrm{d}, \Gamma),\\
K &= \mathbf{t} \circ \widetilde{\Psi}_{\mathrm{DL}} - \frac{1}{2}\mathrm{Id} &&: H_{\perp}^{-1/2}\Lambda^p(\mathrm{d}, \Gamma) \to H_{\perp}^{-1/2}\Lambda^p(\mathrm{d}, \Gamma),
\end{aligned}$$

where indices p have been omitted.

Eventually [1, (42)], [7, (97)] it can be shown that the Dirichlet problem (1) can be cast into an equivalent boundary integral equation, whose weak variational form reads: Find Neumann data $\boldsymbol{\gamma} \in H_{\|}^{-1/2}\Lambda^1(\delta, \Gamma)$ such that

$$b\Big(\boldsymbol{\gamma}', \tilde{V}\boldsymbol{\gamma}\Big) = b\Big(\boldsymbol{\gamma}', (\frac{1}{2}\mathrm{Id} - K)\mathbf{t}\boldsymbol{\omega}_{\mathrm{i}}\Big) \tag{2}$$

holds for all $\boldsymbol{\gamma}' \in H_{\|}^{-1/2}\Lambda^1(\delta, \Gamma)$. Herein, $b(\cdot, \cdot) : H_{\|}^{-1/2}\Lambda^1(\delta, \Gamma) \times H_{\perp}^{-1/2}\Lambda^1(\mathrm{d}, \Gamma) \to \mathbb{C}$ is the sesquilinear form defined in [7, (31)].

The weak variational form (2) leads itself easily to discretization. To this end, we approximate the boundary Γ by a Lipschitz polyhedron and the Neumann data $\boldsymbol{\gamma}$ by piecewise linear continuous functions. Piecewise linear continuous functions yield a conforming discretization of $H_{\|}^{-1/2}\Lambda^1(\delta, \Gamma)$.

Issues.

1. It is expected that there exists a discrete spectrum of wave numbers k that yields exterior or interior resonances, such that the formulation (2) breaks down.

2. The considered wave number has to exceed a certain cutoff threshold, $k > k_{\mathrm{c}}(R)$, for propagating waves to exist on the sphere.

3. Lemma 1 in [7] has to be extended and validated for the case of manifolds that exhibit curvature.

Tasks.

1. Get acquainted with the functional analytic and differential geometric background and setting of this Master Thesis project. All numerical implementations shall be provided as documented MATLAB code.

2. Derive and implement the Helmholtz-Green kernel double forms $G_p(x, y)$, $p = 0, 1$. Additional useful references are [5] and [9, Ch. 2]. A challenge resides in a fast and stable numerical implementation of the hypergeometric kernel functions. Examine the radiation of a point source on the sphere.

3. Derive and implement a Galerkin discretization with piecewise linear elements, starting from the weak variational formulation (2) of the scattering problem. Consider as test case a disc-shaped domain Ω centered in the north pole of the sphere, illuminated by a point source located in the south pole of the sphere.

4. Document the results as a Master Thesis that adheres to scientific standards. Give a presentation to SAM and TU Tampere members about the outcome of the work.

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