Bachelor Thesis Project/ Term Project

(Computational Science & Engineering, Mathematics)

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Impact of highly non-affine, 3D, hybrid meshes on the convergence of Iterative Methods

Prerequisites. Knowledge about the finite element method, familiarity with C++ and iterative methods for linear systems.

Background This project is part of a larger collaboration with ABB Corporate Research which is funded by CTI (Comission for Technology and Innovation). One of its goals is to improve the accuracy of electromagnetic field computations. The resulting linear systems are generally very large and finding an appropriate preconditioner is a challenging task.

Problem description. We consider the following elliptic boundary value problem posed on a three-dimensional domain $\Omega \subset \mathbb{R}^3$:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \, , \\ u &= g \quad \text{on } \Gamma_D \, , \\ \mathbf{n} \cdot \mathbf{grad}\, u &= 0 \quad \text{on } \Gamma_N \, . \end{aligned} \tag{1}$$

Here, u is scalar valued function, and Γ_D and Γ_N are two separated parts of the boundary, $\Gamma_D \cup \Gamma_N = \partial\Omega$ and g specifies the dirichlet data. The boundary value problem (1) has a variational formulation in $H^1(\Omega)$: $\mathsf{a}(u, v) = \ell(v)$ for all $v \in H^1(\Omega)$.

In both cases the discretization is based on a fully hybrid mesh which involves first order tetrahedrons, hexahedrons, prisms as well as pyramids. Let $\mathcal{T}^h$ be such a triangulation of the domain Ω and let $F_K : \tilde{K} \mapsto K$ be the mapping from a particular reference element $\tilde{K}$ to the mesh element K.

For a general hexahedral element F_K is a trilinear mapping which is non-affine. We propose to measure the deviation from an affine mapping by (cf. [4])

$$\gamma = \max_{K \in \mathcal{T}^h} \sup_{\tilde{x} \in \tilde{K}} \left\| \mathbf{B}_K^{-1} \, \mathbf{D}F_K(\tilde{x}) - \mathbf{I} \right\|_2 \tag{2}$$

Here T is an arbitrary mesh element of the triangulation $\mathcal{T}$, $\mathbf{B}_K$ is the Jacobian of F_K at the center of the element and $\mathbf{D}F_K(\tilde{x})$ is the Jacobian at $\tilde{x}$. Note that if all maps F_K are linear (i.e. with a constant Jacobian) then $\gamma = 0$.

Let $\kappa = {}^{\lambda_{\max}}/_{\lambda_{\min}}$ be the *spectral condition number* of the system matrix of problem (1). If κ is large, the system matrix is said to be *ill-conditioned*. Moreover, κ characterizes the eigenvalue spectrum of the matrices and affects the number of iterations which are needed by the preconditioned conjugate gradient method to solve problem (1).

If all mappings F_K are linear, one can give bounds for κ in terms of mesh parameters (see e.g. [1], [3]). In particular one can show that anisotropic meshes (e.g. meshes with elements that have high aspect-ratio) are in general more ill-conditioned than comparable isotropic meshes. However, it can be shown that a diagonal preconditioner remedies the high condition number of an anisotropic mesh (cf. [3]). Also, it has been observed that Incomplete LU (ILU) preconditioners perform very well on anisotropic meshes (see [2]).

Issues. In practice the mappings F_K are often non-linear (e.g. tri-linear for a hexahedral element and rational for the pyramidal element) and it is not clear if bounds similar to the ones presented in [3] still apply. It also not clear if diagonal preconditioner and/or ILU preconditioners remedies the bad condition number of anisotropic meshes.

Tasks. Study the effects of non-linear mappings F_k on the condition number κ and determine experimentally if bounds similar to the ones presented in [1], [3] are still feasible. The non-linearity could be measured with Eq. (2) but other expressions might be even more suitable.

Moreover, it should be analyzed if diagonal respectively ILU preconditioners improve the conditioning of stiffness matrices of anisotropic meshes if the element mappings F_K are non-affine.

The implementation can be done in the HyDi framework which is used by ABB Corporate research. HyDi already provides functionality to assemble the stiffness matrix but a few things would need to be implemented before starting with numerical experiments:

1. Program a simple, parametric mesh generator which creates meshes of arbitrary "non-linearity".
2. Implement a (inverse) power iteration to determine κ from a given stiffness matrix.

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References

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