Bachelor/Master/Diploma Thesis Project

(Mathematics, Computational Science & Engineering)

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Asymptotic Method for Electrostatic Force Calculation

Prerequisites. Knowledge about the finite element method for the numerical solution of elliptic boundary value problems [1], cf. course of numerical solution of elliptic and parabolic partial differential equations.

Problem description. The electrostatic potential in a bounded volume Ω subject to given voltages imposed on connected components of its boundary can be computed by solving

$$-\Delta u = 0 \quad , \quad u = g \quad \text{on } \partial\Omega \; . \tag{1}$$

The total electrostatic force on a connected component Γ of $\partial\Omega$ can be expressed by the functional

$$F = \int_{\Gamma} \left| \frac{\partial u}{\partial \mathbf{n}} \right|^2 \mathrm{d}S \; . \tag{2}$$

Note that F is not continuous on the natural energy space $H^1(\Omega)$, but elliptic regularity theory [3, Thm. 4.24] ensures that for $g \in H^1(\partial\Omega)$ we can expect $\frac{\partial u}{\partial \mathbf{n}} \in L^2(\partial\Omega)$. Nevertheless, using standard Lagrangian finite elements and a direct evaluation of F yields poor results.

Another approach [4] replaces (1) by

$$-\Delta u_\epsilon = 0 \quad \text{in } \Omega \; , \tag{3}$$

$$u_\epsilon + \epsilon \frac{\partial u_\epsilon}{\partial \mathbf{n}} = g \quad \text{on } \Gamma \; , \quad u_\epsilon = g \quad \text{on } \partial\Omega \setminus \Gamma \; . \tag{4}$$

Using integration by parts, we find

$$J(\epsilon) := \int_{\Omega} |\,\mathbf{grad}\, u_\epsilon|^2 \,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} u \frac{\partial u_\epsilon}{\partial \mathbf{n}} \,\mathrm{d}S = \int_{\partial\Omega} g \frac{\partial u_\epsilon}{\partial \mathbf{n}} \,\mathrm{d}S - \epsilon \int_{\partial\Omega} \left| \frac{\partial u_\epsilon}{\partial \mathbf{n}} \right|^2 \mathrm{d}S \; .$$

This means that

$$F = -\frac{d}{d\epsilon} J(u_\epsilon) \; . \tag{5}$$

Formula (5) can be used to evaluate F but computing $J(u_\epsilon)$ for different small values of ϵ and performing extrapolation to zero [2, Sect. 9.4].

Issues. Convergence of extrapolation technqiues and comparison with straightforward finite element implementation of F-evaluation.

Task. MATLAB implementation and numerical investigation of extrapolation technique. Try to provide theoretical underpinning for the choice of ϵ.

References

[1] D. Braess, *Finite Elements*, Cambridge University Press, 2nd ed., 2001.

[2] P. Deuflhard and A. Hohmann, *Numerische Mathematik I*, DeGruyter, Berlin, 3 ed., 2002.

[3] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, UK, 2000.

[4] M. Raffy, *Sur l'approximation du flux d'energie rayonne a traverse une enceinte*, phd thesis, Universite L. Pasteur, Strasbourg, France, 1975.