Goals:

- Devise space-time DG-method for *the wave equation*:

$$u_{tt} - u_{xx} = f \qquad \text{in } Q := \Omega\times]0, T[$$

- Implement method and test it
- Investigate stabilty of method

Weak formulation:

- $\mathcal{M} = \{K\}$ is a mesh that covers the space-time domain Q
- Testfunction $v \in V_h(\mathcal{M}) = \bigotimes_{K\in\mathcal{M}} P_p(K)$
- Notation: $\lozenge u = (u_x, -u_t)$ and $\nabla u = (u_x, u_t)$.

$$-\sum_{K\in\mathcal{M}} (\nabla \cdot \lozenge u, v)_K \,\mathrm{d}\boldsymbol{x} = \sum_{K\in\mathcal{M}} (f, v)_K \,\mathrm{d}\boldsymbol{x} \qquad \overset{\text{i.b.p.}}{\Rightarrow}$$

$$\sum_{K\in\mathcal{M}} (\lozenge u, \nabla v)_K - \sum_{K\in\mathcal{M}} \langle \lozenge u \cdot n, v\rangle_{\partial K} = \sum_{K\in\mathcal{M}} (f, v)_K \qquad \overset{\text{DG-magic}}{\Rightarrow}$$

$$\sum_{K\in\mathcal{M}} (\Diamond u, \nabla v)_K - \langle\{\Diamond u\},[v]\rangle_{\mathcal{E}} - \langle\{\Diamond v\},[u]\rangle_{\mathcal{E}} = \sum_{K\in\mathcal{M}} (f,v)_K$$

Notation: $\{v\} = (v^+ + v^-)/2$ and $[v] = v^+ n^+ + v^- n^-$.
Numerical scheme is then:

$\forall\, K \in \mathcal{M}$, seek $u_h \in V_h$ such that

$$a_K(u_h, v_h) = \ell_K(v_h), \qquad \text{for all } v_h \in P_p(K)$$

where $\ell_K(v) = (f,v)_K$ and

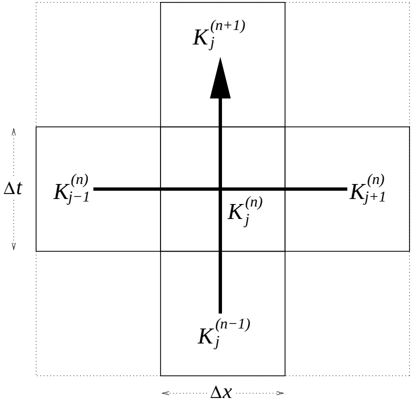
$$a_K(u,v) = (\Diamond u, \nabla v)_K - \langle\{\Diamond u\},[v]\rangle_{\partial K} - \langle\{\Diamond v\},[u]\rangle_{\partial K} + \langle\alpha[u],[v]\rangle_{\partial K}$$

Choose basis, and plug in:

$$a_K(\sum_i \mu_i b_i, \sum_j q_j b_j) = \ell_K(\sum_j q_j b_j), \quad \text{for all } v_h = \sum_j q_j b_j \overset{\text{local b.f.}}{\Rightarrow}$$

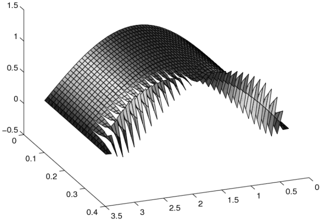
$$\vec{\mu}_j^{(n+1)} = \left(A_j^{(n+1)}\right)^{-1} \left(\vec{\ell} - A_j^{(n-1)} \vec{\mu}_j^{(n-1)} - \sum_{i=j-1}^{j+1} A_i^{(n)} \vec{\mu}_i^{(n)}\right)$$

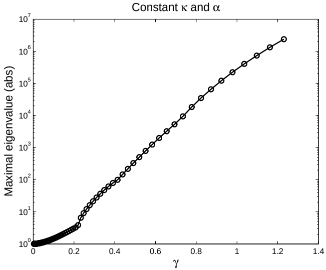
- Possible if we choose *locally* supported basis functions
- Now we have *explicit* scheme, suited for implementation



- Determined entries in matrices analytically in Maple
- Then implemented method in Matlab

- Unfortunately:





- Numerical experiments: Blow up in solutions
- Von Neumann analysis showed *unconditional* instability
- Possible remedy? Perhaps: Use mesh that avoids vertical edges