Local Multi-trace Boundary Element Formulation for Diffusion Problems

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- Diffusion equation

$$-\mu_i \Delta U(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega_i$$

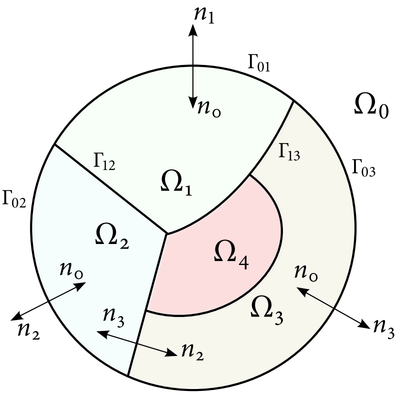
- Transmission conditions across adjacent subdomains

$$U_i - U_j = 0 \quad \text{on } \Gamma_{ij}$$

$$\mu_i \frac{\partial U_i}{\partial \mathbf{n}_i} + \mu_j \frac{\partial U_j}{\partial \mathbf{n}_j} = 0 \quad \text{on } \Gamma_{ij}$$

- Decay conditions at ∞
- Inhomogeneous Dirichlet boundary conditions on Γ_{DIR}

$$U|_{\Gamma_{DIR}} = g$$

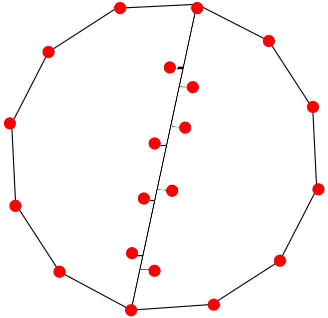


Composite computational domain

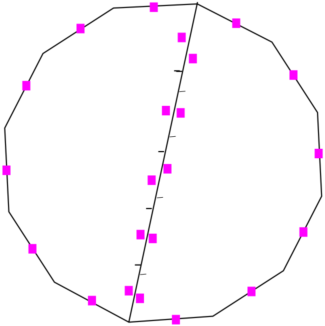
Sources are introduced by a given "incoming wave" U_{inc}, satisfying $-\mu_0 \Delta U_{inc} = 0$, on the exterior domain.

Multi-trace space

Multi-trace space : $\mathbb{V}_1 \times \cdots \times \mathbb{V}_N$



p.w. linear C^0 BEM
$\subset H^{\frac{1}{2}}(\partial\Omega_1) \times H^{\frac{1}{2}}(\partial\Omega_2)$



p.w. constants $\subset H^{-\frac{1}{2}}(\partial\Omega_1) \times H^{-\frac{1}{2}}(\partial\Omega_2)$

Local multi-trace formulation

$$\begin{aligned}(A_0 - \tfrac{1}{2}\,\mathrm{Id})\gamma^0 U &= 0\\ (A_1 - \tfrac{1}{2}\,\mathrm{Id})\gamma^1 U &= 0\\ (A_2 - \tfrac{1}{2}\,\mathrm{Id})\gamma^2 U &= 0\end{aligned} \;\Rightarrow\; \begin{aligned} A_0\,\gamma^0 U - \tfrac{1}{2}X_{01}\,\gamma^1 U - \tfrac{1}{2}X_{02}\,\gamma^2 U &= 0\\ -\tfrac{1}{2}X_{10}\,\gamma^0 U + A_1\,\gamma^1 U - \tfrac{1}{2}X_{12}\,\gamma^2 U &= 0\\ -\tfrac{1}{2}X_{20}\,\gamma^0 U - \tfrac{1}{2}X_{21}\,\gamma^1 U + A_2\,\gamma^2 U &= 0\end{aligned}$$

Transmission operators X_{ij}: traces on $\partial\Omega_j \to \Gamma_{ij}$

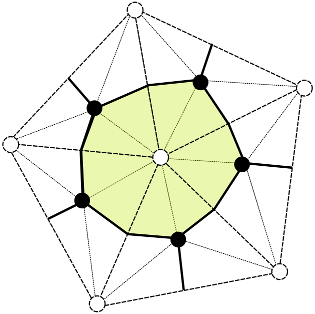
1. Restrict traces on $\partial\Omega_j$ to Γ_{ij}
2. Keep Dirichlet data, flip sign of Neumann data
3. Extend by zero from Γ_{ij} to $\partial\Omega_i$

System matrix

$$
\begin{pmatrix}
A_0 & X_{0,1} & \cdots & & & \cdots & X_{0,N} \\
X_{1,0} & A_1 & X_{1,2} & \cdots & & \cdots & X_{1,N} \\
\vdots & \ddots & \ddots & \ddots & & & \vdots \\
& & & & & & \\
\vdots & & & & & & X_{N-1,N} \\
X_{N,1} & \cdots & & & \cdots & X_{N,N-1} & A_N
\end{pmatrix} \quad (1)
$$

Diagonal: Dense matrices A_i, arising from standard Galerkin BEM discretization in lowest order boundary elements, i.e. p.w. linear C^0-BEM × p.w. constants.
Off-diagonals: transmission matrices X_{ij} (sparse) and $X_{ij} = 0$ when there is no common interface.
It is called **Local** multi-trace because the coupling across subdomains is through sparse matrices and not through BIO's (global multi-trace).

Caldéron preconditioner



Dual mesh (barycentric refinement), dual cell (green), dual nodes (filled black circles)

Caldéron preconditioner

$$P = D^{-T} A_i D^{-1}$$

Stable discrete duality pairing required.

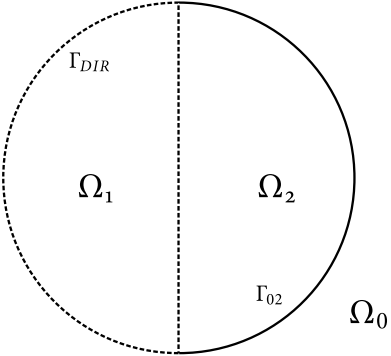
mesh $\mathcal{M}$	$\leftrightarrow$	dual mesh $\widehat{\mathcal{M}}$
nodes	$\leftrightarrow$	cells
edges	$\leftrightarrow$	edges
cells	$\leftrightarrow$	nodes

Dirichlet boundary conditions

Test, trial funtions $\widetilde{H}^{\frac{1}{2}}(\partial\Omega_i \setminus \Gamma_{DIR}) \times H^{-\frac{1}{2}}(\partial\Omega_i)$
$U = g$ on $\Gamma_{DIR} = \partial\Omega_1$

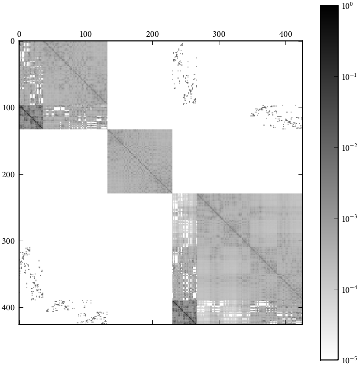
$$\begin{bmatrix} A_0 & & -\frac{1}{2}X_{2,0} \\ & V_1 & \\ -\frac{1}{2}X_{0,2} & & A_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \widetilde{u}_0^D \\ u_0^N \end{bmatrix} \\ u_1^N \\ \begin{bmatrix} \widetilde{u}_2^D \\ u_2^N \end{bmatrix} \end{bmatrix} = r.h.s.$$

Transmission operators $X_{*,1}$ drop out.

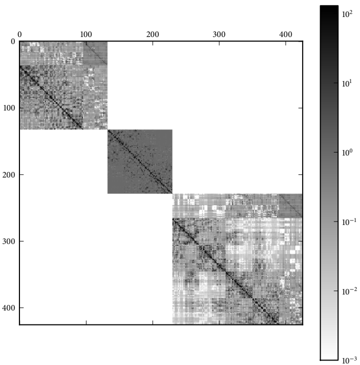


Sparsity patterns

2 subdomains, Dirichlet boundary conditions



System matrix

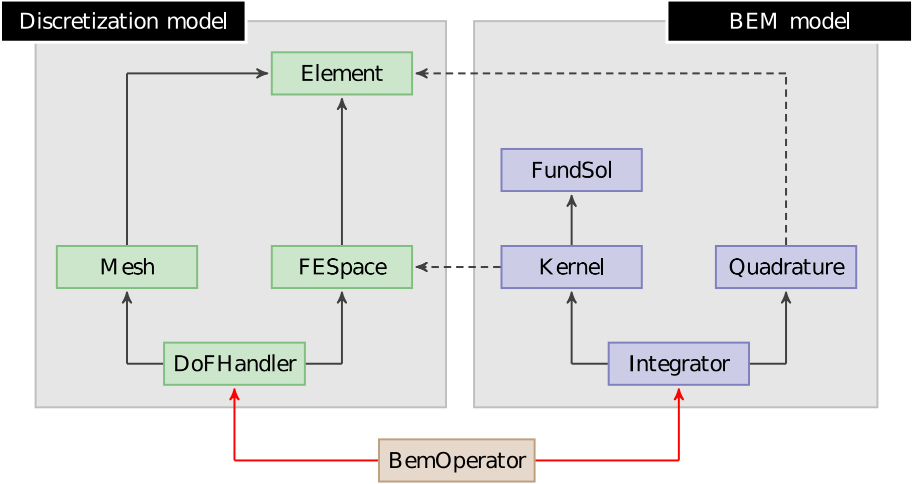


Caldéron preconditioner

The BETL library

- BETL, a C++ template library, developed by Dr. Lars Kielhorn
- Assembly of the discrete boundary integral operators
- Provides interface to AHMED (Adaptive Cross Approximation, M. Bebendorf)
- Shared memory parallelism through AHMED and the multi-threaded Intel MKL BLAS routines
- Matrix expression templates for Matrix-Matrix and Matrix-Vector operations
- Interface to Umfpack, SuperLU, etc

$$A[i,j] = \int_{supp(\phi_i)} \phi_i(\mathbf{x}) \int_{supp(\psi_j)} G(\mathbf{y}-\mathbf{x})\psi_j(\mathbf{y})\, \mathrm{d}s_y\, \mathrm{d}s_x$$



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Tasks

- Assembly of the system matrix and the Calderéron preconditioner, i.e. implementing their matrix-vector products.
- Inclusion of Dirichlet boundary conditions
- Load vector

Caldéron preconditioner $P = D^{-T} A_i D^{-1}$

BEM matrices

$$A_{\{\widehat{D},\widehat{N}\},\{\widehat{D},\widehat{N}\}} := \begin{pmatrix} C^T_{\widetilde{N},\widehat{N}} & \\ & C^T_{\widetilde{D},\widehat{D}} \end{pmatrix} \begin{pmatrix} -K_{\widetilde{N},\widetilde{D}} & V_{\widetilde{N},\widetilde{N}} \\ W_{\widetilde{D},\widetilde{D}} & K'_{\widetilde{D},\widetilde{N}} \end{pmatrix} \begin{pmatrix} C_{\widetilde{D},\widehat{D}} & \\ & C_{\widetilde{N},\widehat{N}} \end{pmatrix}.$$

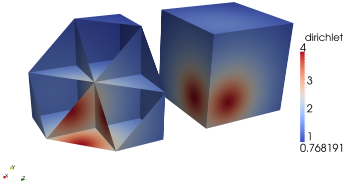
Basis transformation D

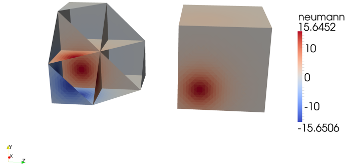
LU decomposition via UMFPACK

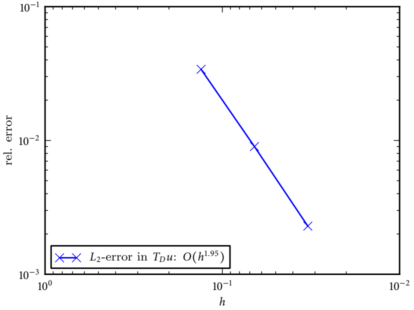
Expression templates

```
y += (diag_matrix(invMN^'T',invMD^'T')*
      diag_matrix(CBN^'T', CBD^'T')*C*
      diag_matrix(CBD,CBN)*
      diag_matrix(invMD,invMN))*x
```

Unit cube, 8 subdomains, Dirichlet boundary condition $g = \frac{1}{\|\mathbf{x}\|}$ on $\partial\Omega_1$.





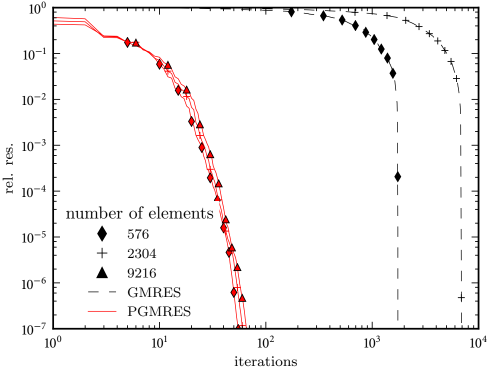


L_2-error in $u = \mathcal{O}(h^2)$, h: mesh width

#Elements: 9216

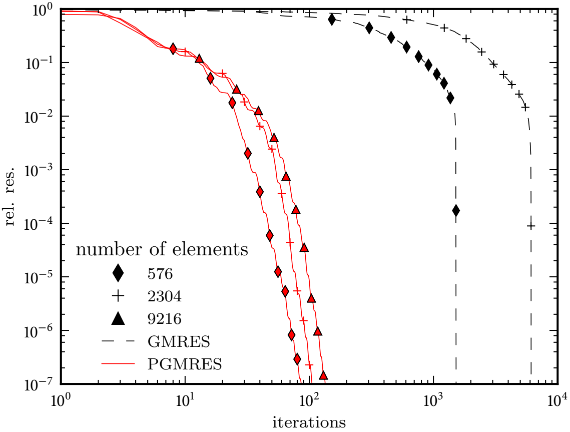
GMRES convergence

Unit cube, 8 subdomains



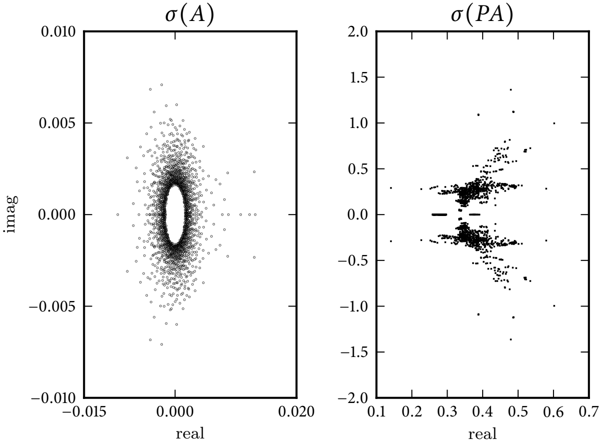
Preconditioned system: #iterations $=O(h^{-0.1})$
Unpreconditioned system: #iterations $= O(h^{-2})$

Unit cube, 8 subdomains, Dirichlet boundary conditions

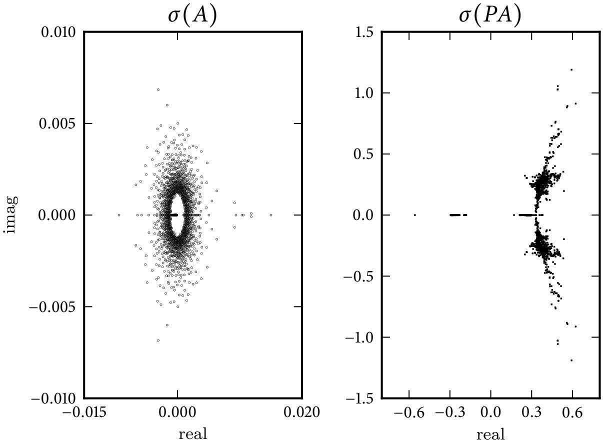


Preconditioned system: #iterations $=O(h^{-0.3})$
Unpreconditioned system: #iterations $= O(h^{-2})$

Eigenvalue distributions



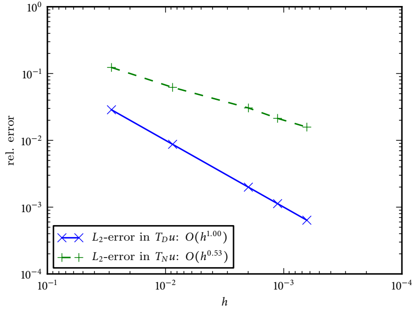
Unit cube, 8 subdomains, 576 elements



Unit cube, 8 subdomains, Dirichlet BC, 576 elements

Diffusion

Analytical solution for a sphere via spherical harmonics.

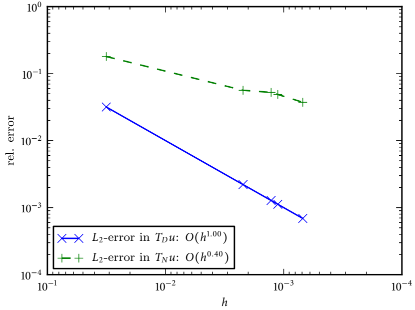


Convergence rates, unit sphere (2 subdomains)

$$\mu(\mathbf{x}) = \begin{cases} 1.5 & \|\mathbf{x}\| > 1 \\ 1 & \|\mathbf{x}\| \leq 1 \end{cases}$$

L_2-errors

$T_D U = \mathcal{O}(h)$
$T_N U = \mathcal{O}(h^{0.53})$

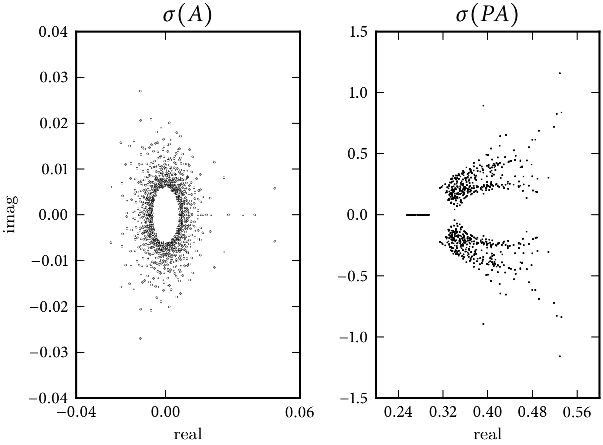


Convergence rates, unit sphere (2 subdomains)

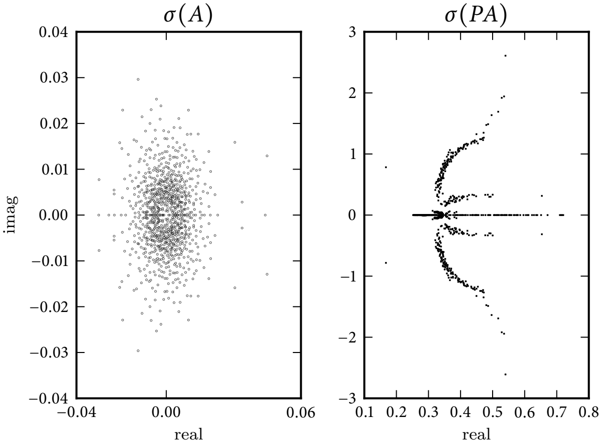
$$\mu(\mathbf{x}) = \begin{cases} 10 & \|\mathbf{x}\| > 1 \\ 1 & \|\mathbf{x}\| \leq 1 \end{cases}$$

L_2-errors

$$T_D U = \mathcal{O}(h)$$
$$T_N U = \mathcal{O}(h^{0.4})$$


σ(A)
0.04
0.03
0.02
0.01
0.00
−0.01
−0.02
−0.03
−0.04
imag
−0.04
0.00
0.06
real
σ(PA)
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.24
0.32
0.40
0.48
0.56
real

$$\mu_0 = 1.5$$

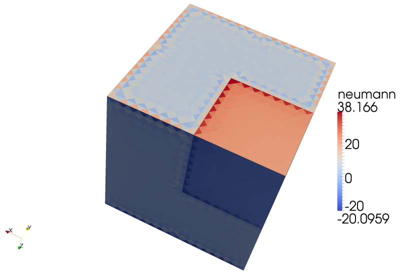

σ(A)
0.04
0.03
0.02
0.01
0.00
−0.01
−0.02
−0.03
−0.04
imag
−0.04
0.00
0.06
real
σ(PA)
3
2
1
0
−1
−2
−3
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
real

$\mu_0 = 10$

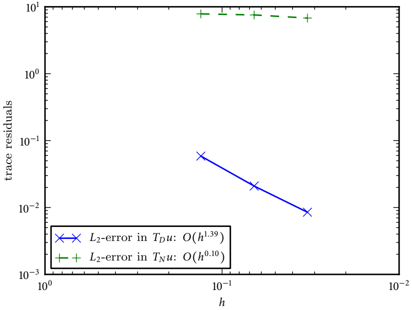
Diffusion, cube (8 subdomains)

Trace residuals

$T_D U = \mathcal{O}(h^{1.4})$
$T_N U = \mathcal{O}(h^{0.1})$



$T_N U$, # elements 9216



Convergence rates, unit cube (8 subdomains)

- Jumps in the diffusion constants produce matrices with high condition numbers.
- Neumann data converges very slowly.
- Preconditioned GMRES requires approx. two times more iterations, but is still independent of the mesh width.

Summary

- Verification of the code by problems where an analytical solution is known.
- GMRES iterations for the preconditioned system are independent of the number of unkowns,
- and almost independent when Dirichlet boundary conditions are present.
- Code is generic for Helmholtz and Laplace problems.