FINDING WEAKLY EQUIVALENT MINIMAL SUBCOMPLEXES BY CELL DELETIONS

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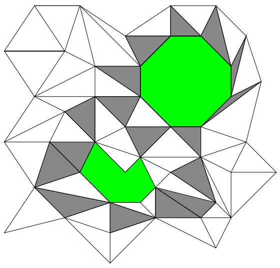
ABSTRACT. This article discusses local criteria for homologically redundant cells and thus introduces an algorithm to find a quasi-minimal subcomplex of a simplicial complex of the same homology type.



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1. Introduction

Given a simplicial complex $\mathfrak{M}$ as a triangulation of a topological space Ω. In order to compute the homology groups and its generators one has to compute the kernel and image of the cell-incidence matrices or boundary maps ∂_n. This process is computationally expensive if said matrices are large. Hence it would be convenient to simplify the complex by reducing the number of cells. Direct approaches include edge contractions and cell deletions. The first method was analyzed in [2] where local criteria like "link conditions" were introduced. In this article we focus on the latter method and analyze its properties.

1.1. **Range of Application.** The criteria for valid cell deletions depend on the boundaries of a simplex and its neighbors. Our algorithm is not applicable, if all n-simplices have all $n+1$ of their faces in common with other n-simplices, since every cell deletion would result in a hole. A simplicial complex which is already minimal in some sense displays the same problem. We give two examples for boundaryless and quasi-minimal simplicial complexes:

i. Any triangulation of a n-dimensional torus $\Omega = \mathbb{T}^n \subset \mathbb{R}^{n+1}$ has no boundary-adjacent cells of dimension $n-1$ since $\partial\Omega = \emptyset$. Any cell deletion would create a hole.
ii. A chain of n-simplices, each one attached to its neighbors at two vertices, forming a closed cycle is quasi-minimal: any cell deletion would disconnect the cycle.

Despite of these examples there are many situations where our algorithm provides a good simplification of a triangulation.

1.2. **Edge Contractions versus Cell Deletions.** Both edge contractions and cell deletions depend on local criteria involving neighbors of a simplex. The advantage of edge contractions is that meshes without boundary are reducible. The advantage of cell deletions is that the resulting complex is a subcomplex of the original one - no information about the embedding is lost and the number of neighbors of a simplex never increases.

2. Definitions

2.1. **Preliminaries.** Let V be a discrete set. The elements of V are referred to as *vertices* or 0-*simplices*. For computational reasons we demand V to be finite.

> **Definition 2.1** (Simplex)**.** We call two $(n+1)$-tuples s, t of elements of V equivalent, if there is a permutation p of the elements of s, such that $p(s) = t$ and $\operatorname{sign}(p) = 1$. A *$n$-simplex* τ is an equivalence class of such a $(n+1)$-tuple. We write $\tau = (v_0, \ldots, v_n) = \operatorname{sign}(p) \cdot (v_{p(0)}, \ldots, v_{p(n)})$.
> We refer to $\operatorname{sign}(p)$ as the relative *orientation* of a simplex.

Remark. Concerning logical set operations like $\cap$, $\cup$, $\subset$ and $\in$, we handle simplices like sets without respect to orientation.

To understand the above definition better, we give an example. The simplex (v_0, v_1, v_2) holds the following identities:

· $(v_0, v_1, v_2) = -1 \cdot (v_1, v_0, v_2) = 1 \cdot (v_1, v_2, v_0) \neq (v_1, v_0, v_2)$ etc.
· $(v_0, v_1) \subset (v_0, v_1, v_2)$, $(v_1, v_0) \subset (v_0, v_1, v_2)$, $v_2 \in (v_0, v_1, v_2)$ etc.

This means, "=" pays respect to the orientation, but "$\subset$" does not.

Definition 2.2 (Subsimplex and Face)**.** A simplex σ is called a subsimplex or face of a simplex τ, if $\sigma \subset \tau$.

Remark. Geometrically, simplices are defined as the convex hull of their vertices. The above definition comes in handy for our application, because of the naturally resulting orientation of a simplex. However, the topological definition is more intuitive and enlightening.

Definition 2.3 (Inherited Orientation $\mathfrak{o}_\tau$)**.** Let τ be a n-simplex and σ a $(n-1)$-subsimplex. The inherited orientation $\mathfrak{o}_\tau(\sigma)$ of σ is the sign of the permutation p of the vertices of τ, such that the first n entries of $p(\tau)$ coincide with σ. For a $(n-2)$-subsimplex η we define the relative orientation as $\mathfrak{o}_\tau(\eta) = \mathfrak{o}_\sigma(\eta) \cdot \mathfrak{o}_\tau(\sigma)$ where $\eta \subset \sigma \subset \tau$ and σ is an $(n-1)$-simplex. Proceed analogously for lower dimensional subsimplices.

> **Definition 2.4** (The Split Map $\mathfrak{d}^l_n$)**.** Let σ be a n-simplex, and let the set $\mathcal{S}^l$ contain *all possible* l-simplices with vertices in V. We define the split map $\mathfrak{d}^l_n : \mathcal{S}^n \to 2^{\mathcal{S}^l}$ as
> $$\mathfrak{d}^l_n(\sigma) = \{\tau \in \mathcal{S}^l | \tau \subset \sigma \ \wedge \ \mathfrak{o}_\sigma(\tau) = +1\}$$
> In other words, $\mathfrak{d}^l_n(\sigma)$ is the set of all l-subsimplices of σ.

Remark. $\mathfrak{d}_l^l(\sigma) = \{\sigma\}$, $\forall \sigma \in \mathcal{S}^l$ and $\mathfrak{d}_l^{l+k}(\sigma) = \emptyset$, $\forall k > 0$.

Definition 2.5 (Simplicial Complex)**.** Let S_k be a collection of k-simplices, $k \in \{1, \ldots, m\}$, $S_0 = V$. To prevent ambiguity, we demand S_k to contain either σ or $-1 \cdot \sigma$. We call the set $\mathfrak{M} := \{S_0, \ldots, S_m\}$ a simplicial m-complex if

- $\forall l > 0 :\ \forall \sigma \in S_l : \forall \eta \in \mathfrak{d}_l^{l-1}(\sigma)$: either $\eta \in S_{l-1}$ or $-1 \cdot \eta \in S_{l-1}$.
- $\forall l > 0 :\ \forall \sigma, \tau \in S_l :\ n := max\big(\{k < l | \mathfrak{d}_l^k(\sigma) \cap \mathfrak{d}_l^k(\tau) \neq \emptyset\} \cup \{-1\}\big)$, either $n = -1$ or $|\mathfrak{d}_l^n(\sigma) \cap \mathfrak{d}_l^n(\tau)| = 1$.

This implies that any face of a simplex is part of the complex and the intersection of two simplices is face of both of them.
A simplicial complex is called to be of *homogeneous dimension* m if S_m is non-empty, $S_{m+1} = \emptyset$ and $\forall \tau \in S_l : \exists \sigma \in S_m :\ \tau \in \mathfrak{d}_m^l(\sigma)$.

Remark. When using the topological definition of simplices, the above definition becomes easier to comprehend, but again, it does not offer handling of orientation.

Definition 2.6 (Subcomplex)**.** Let $\mathfrak{K} = \{\tilde{S}_0, \ldots, \tilde{S}_m\}, \mathfrak{M} = \{S_0, \ldots, S_m\}$ be two simplicial complexes. We call $\mathfrak{K}$ a subcomplex of $\mathfrak{M}$ if $\forall n \in \{0, \ldots, m\} :\ \tilde{S}_n \subset S_n$ and write $\mathfrak{K} \subset \mathfrak{M}$.

Definition 2.7 (Complex Expansion $\uplus$)**.** Given a simplicial complex $\mathfrak{M}$ and a k-simplex $\eta \notin \mathfrak{M}$. We want to expand $\mathfrak{M}$ by η and all of its subsimplices and vertices:

$$\mathfrak{M} \uplus \eta := \big\{S_0 \cup \mathfrak{d}_k^0(\eta), \ldots, S_k \cup \mathfrak{d}_k^k(\eta), S_{k+1}, \ldots, S_m\big\}$$

Note that the resulting complex possibly is not of homogeneous dimension.

Definition 2.8 (Homogenization $\mathcal{H}_n$ of a Complex)**.** Let $\mathfrak{M} = \{S_0, \ldots, S_m\}$ be a simplicial complex (not necessarily of homogeneous dimension) and $n \in \mathbb{N}$. A morphism $\mathcal{H}_n$ in the category of simplicial complexes is called a homogenization of $\mathfrak{M}$, if $\mathcal{H}_n(\mathfrak{M})$ is a simplicial complex of homogeneous dimension n, $\mathfrak{M} \subset \mathcal{H}_n(\mathfrak{M})$ and if the first $m+1$ homology groups of $\mathfrak{M}$ and $\mathcal{H}_n(\mathfrak{M})$ are isomorphic and $H_k\big(\mathcal{H}_n(\mathfrak{M}), \mathbb{Z}\big)$ is trivial, $\forall k > m$.

It is not clear that a homogenization of a simplicial complex can be realized easily on a machine. Therefore we give a direct method of how to compute $\mathcal{H}_n(\mathfrak{M})$:

1. given: $\mathfrak{M} = \{S_0, \ldots, S_m\}$, $n \geq m$.
2. $\mathfrak{M} \mapsto \{S_0, \ldots, S_m, S_{m+1}, \ldots, S_n\}$ where $S_k = \emptyset$, $\forall k > m$
3. for $k = m$ to 0 (step -1)
   do
   3.1 for each $\tau = (v_0, \ldots, v_{k-1}) \in S_k$ with $\nexists \sigma \in S_n :\ \tau \subset \sigma$:
      31.1 let $w_1, \ldots, w_{n-k} \notin S_0$ be *new vertices*
      31.2 $\mathfrak{M} \mapsto \mathfrak{M} \uplus (v_0, \ldots, v_{k-1}, w_1, \ldots, w_{n-k})$
4. next k

Definition 2.9 (k-Skeleton Skel_k)**.** Let $\mathfrak{M} = \{S_0, \ldots, S_m\}$ be a simplicial complex and $k < m$. The k-skeleton of $\mathfrak{M}$ is the subcomplex $\mathrm{Skel}_k(\mathfrak{M}) := \{S_0, \ldots, S_k\}$

2.2. **Cell Deletions.** From now on, the simplicial complex $\mathfrak{M}$ is to be of homogeneous dimension m and the space $\Omega := |\mathfrak{M}|$ has a non-empty boundary $\partial\Omega$.

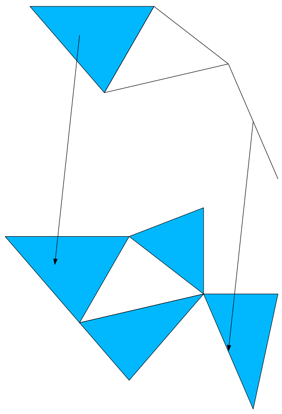


FIGURE 1. Example for $\mathcal{H}_2$

Definition 2.10 (Cell Deletion $\mathcal{D}_\tau$)**.** Let $\mathfrak{M} = \{S_0, \dots, S_m\}$ be a simplicial complex, S_l its sets of l-simplices. Let $\mathcal{S}^n$ be the set of *all possible* n-simplices with vertices in S_0, $\tau \in S_m$ be a simplex of maximal dimension. We define k-cell deletions $\mathcal{D}^k_\tau : 2^{\mathcal{S}^k} \to 2^{\mathcal{S}^k}$ as

$$\mathcal{D}^k_\tau(S) := \left[S \setminus \mathfrak{d}^k_m(\tau)\right] \cup \bigcup_{\sigma \in S_m \setminus \{\tau\}} \mathfrak{d}^k_m(\sigma)$$

for any subset S of $\mathcal{S}^k$.
Now define $\mathcal{D}_\tau(\mathfrak{M}) := \{\mathcal{D}^0_\tau(S_0), \dots, \mathcal{D}^m_\tau(S_m)\}$.

Remark. A cell deletion $\mathcal{D}_\tau$ can be considered as the inverse of a complex expansion $\cdot \uplus \tau$. Furthermore, $\mathfrak{M} \uplus \sigma$ is the smallest simplicial complex $\mathfrak{N}$ containing σ with $\mathfrak{M} \subset \mathfrak{N}$ and $\mathcal{D}_\tau(\mathfrak{M})$ is the biggest subcomplex of $\mathfrak{M}$ of homogeneous dimension not containing τ.

Lemma 2.11. *If $S_m \neq \{\tau\}$ then $\mathcal{D}_\tau(\mathfrak{M})$ is a simplicial complex, even a subcomplex of $\mathfrak{M}$ of the same homogeneous dimension.*

Proof. $\mathcal{D}^m_\tau(S_m)$ contains all m-simplices but τ. For each of them, its faces are contained in $\mathcal{D}^{m-1}_\tau(S_{m-1})$, because of the definition of $\mathcal{D}^{m-1}_\tau$. Concerning the intersection of two simplices, the properties of $\mathfrak{M}$ are invariant under $\mathcal{D}_\tau$ except for those intersections involving τ, which is again no member of $\mathcal{D}_\tau(\mathfrak{M})$.
As all the faces of τ have been removed, if they were not boundaries of a $\sigma \neq \tau$, $\mathcal{D}_\tau(\mathfrak{M})$ is of homogeneous dimension. ☐

Definition 2.12 (Valid Cell Deletions)**.** A cell deletion $\mathcal{D}_\tau$ on a simplicial complex $\mathfrak{M}$ is called *valid*, if all homology groups of $\mathcal{D}_\tau(\mathfrak{M})$ and $\mathfrak{M}$ are isomorphic: $H_k(\mathfrak{M}) \cong H_k(\mathcal{D}_\tau(\mathfrak{M}))$. In this case, we call τ "flesh", otherwise "bone".

Now we are trying to find a method, to distinguish flesh from bones without having to compute the entire homology groups. We will do this by analyzing local properties of the triangulation and we check, whether the removal of τ and its faces would somehow disconnect $|\mathfrak{M}|$.

Definition 2.13 (Relevant Boundary $\overline{\mathfrak{d}_k^l}$)**.** Define the relevant l-boundary $\overline{\mathfrak{d}_k^l}$: $S_k \to 2^{S^l}$ as

$$\overline{\mathfrak{d}_k^l}(\tau) = \mathfrak{d}_k^l(\tau) \cap \{\sigma \in S_l | \exists \eta \in S^k : \sigma \subset \eta,\ \eta \neq \tau\}$$

In other words, the relevant boundary are those subsimplices which are also boundaries of other simplices.

The following criteria are special cases for 2 and 3 dimensions.

Theorem 2.14. *Let $m = 2$. Then τ is flesh if one of the following is true*

i. $\sharp\big(\overline{\mathfrak{d}_2^1}(\tau)\big) = 2$
ii. $\sharp\big(\overline{\mathfrak{d}_2^1}(\tau)\big) \in \{0,1\}$ *and* $\sharp\big(\overline{\mathfrak{d}_2^0}(\tau)\big) = 1 + \sharp\big(\overline{\mathfrak{d}_2^1}(\tau)\big)$

Let $m = 3$. Then τ is flesh if one of the following is true

i. $\sharp\big(\overline{\mathfrak{d}_3^2}(\tau)\big) = 3$
ii. $\sharp\big(\overline{\mathfrak{d}_3^2}(\tau)\big) = 2$ *and* $\sharp\big(\overline{\mathfrak{d}_3^1}(\tau)\big) = 5$
iii. $\sharp\big(\overline{\mathfrak{d}_3^2}(\tau)\big) = 1$ *and* $\sharp\big(\overline{\mathfrak{d}_3^1}(\tau)\big) \in \{3,4\}$ *and* $\sharp\big(\overline{\mathfrak{d}_3^0}(\tau)\big) = \sharp\big(\overline{\mathfrak{d}_3^1}(\tau)\big)$
iv. $\sharp\big(\overline{\mathfrak{d}_3^2}(\tau)\big) = 0$ *and* $\sharp\big(\overline{\mathfrak{d}_3^1}(\tau)\big) \in \{0,\dots,3\}$ *as well as* $\sharp\big(\bigcup_{\sigma \in \overline{\mathfrak{d}_3^1}(\tau)} \mathfrak{d}_1^0(\sigma)\big) = \sharp\big(\overline{\mathfrak{d}_3^0}(\tau)\big)$

Proof. If a n-simplex τ is connected to the rest of the complex via the relevant boundary $\mathfrak{K} = \{\overline{\mathfrak{d}_m^0}(\tau), \dots, \overline{\mathfrak{d}_m^{m-1}}(\tau)\}$, and the topological realization $|\mathfrak{K}|$ is homotopy equivalent to a point, the above criteria state that the cell deletion $\mathcal{D}_\tau$ is valid: The criteria display a representation of all possible numbers of vertices, edges (and if $m = 3$ also triangles) describing subcomplexes of $\mathrm{Skel}_{m-1}(\emptyset \uplus \tau)$ which have a topological realization with trivial homotopy.
If $|\mathfrak{K}|$ is homotopy equivalent to a point, it can be retracted. The simplex τ itself is, as the convex hull of its vertices, also retractible to this point. After this procedure one can blow up the point again to $|\mathfrak{K}|$. This three homotopies realize D_τ. Note that homotopy equivalence implies weak equivalence. ☐

2.3. **Criteria for the General Case ($m > 3$).** All attempts of giving criteria for flesh depending on the number of certain simplices in the relevant boundary remained fruitless. However, there are two possibilities of determining whether a deletion in a higher dimensional simplicial complex is valid:

1. Compute the Betti numbers β_i of the relevant boundary $\{\overline{\mathfrak{d}_m^0}(\tau), \dots, \overline{\mathfrak{d}_m^{m-1}}(\tau)\}$ and determine whether $\beta_i = \delta_{i0}$ (and thus whether the relevant boundary is homologically equivalent to a point); then τ is flesh.
2. Compute the homogenization $\mathfrak{K} := \mathcal{H}_{m-1}\big(\{\overline{\mathfrak{d}_m^0}(\tau), \dots, \overline{\mathfrak{d}_m^{m-1}}(\tau)\}\big)$ of the relevant boundary and apply the cell deletion algorithm (which is introduced in the next

section) on $\mathfrak{K}$. If nothing remains but a $(m-1)$-simplex, τ is flesh. However, for $m > 4$, this process has to be iterated $m-3$ times (for every dimension once).

3. The Algorithm

We will subsequently perform valid cell deletions on $\mathfrak{M}$, $\mathcal{D}_\tau(\mathfrak{M})$, $\mathcal{D}_\sigma\big(\mathcal{D}_\tau(\mathfrak{M})\big)$ etc., until all m-simplices are bones. Note that the resulting bone constellation depends on the chosen order of cell deletions.

Definition 3.1 (Quasi-minimal Subcomplexes)**.** A subcomplex $\mathfrak{K} \subset \mathfrak{M}$ is quasi-minimal, if all simplices of maximal dimension are bones.

Remark. Finding the order of cell deletions which actually produces a minimum (concerning the number of simplices) among the quasi-minimal subcomplexes is an NP complete problem. In most applications this does not matter as even the worst case quasi-minimal subcomplex displays a great simplification of a complex.

Remark. The set of all bones is not a skeleton. The only connection between these expressions is that every $m-k$-skeleton of a complex of homogeneous dimension m is a collection of bones, if $k > 0$.

This is a pseudo-code representation of how the algorithm might be implemented:

1. Given:
   - m the dimension of the simplicial complex
   - S_i the sets of i-simplices, $i \in \{0, \ldots, m\}$
   - $\mathfrak{d}^l_k$ the boundaries of every simplex
   - C_m a set of criteria for valid m-cell deletions
2. Pick a $(m-1)$-simplex σ with $\sigma \notin \overline{\mathfrak{d}^{m-1}_m}(\tau)$, $\forall \tau \in S_m$ (if no such σ exists, stop).
3. Pick a $\tau \in S_m$ with $\sigma \in \mathfrak{d}^{m-1}_m(\tau)$.
4. If τ is flesh according to C_m then $\mathfrak{M} \mapsto \mathcal{D}_\tau(\mathfrak{M})$
5. If the complex is not quasi-minimal, goto 2, else stop.

We did not pay attention in which order we deleted flesh but it is possible to improve either efficiency or quality of the algorithm by modifying the deletion order. When we measured the speed of the algorithm we chose a random order; the result was an apropriate compromise between efficiency and quality.

3.1. **Complexity Analysis.**

3.1.1. *A single cell deletion for $m = 4$.* We first have to determine the relevant boundary of a 4-simplex τ. Note that $\left|\overline{\mathfrak{d}^k_m}(\tau)\right| \le \binom{m}{k}$. The complexity of the homogenization process is of order $O\left((m-2)\cdot\binom{m}{\left(\frac{m}{2}\right)}\right)$ and returns a simplicial complex with at most $\binom{m}{\left(\frac{m}{2}\right)}$ $(m-1)$-simplices. Since $m-1 = 3$ we have criteria for valid cell deletions and are able to predict in $O(n^{1.05})$ (as observations have shown for small complexes) whether τ is flesh.

3.1.2. $m > 4$. As we only have criteria for 3-simplex deletions, we need to recurse the homogenization/simplification process $m-3$ times. Therefore the complexity of our algorithm grows exponentially with the dimension of the simplicial complex. But given a fixed dimension m, the recursion only results in a larger scalar $g(m)$: If the complexity of the 3-dimensional simplification algorithm is $O\big(f(n_3)\big)$, the m-dimensional case is $O\big(g(m)\cdot f(n_m)\big)$ where n is the number of m-simplices.

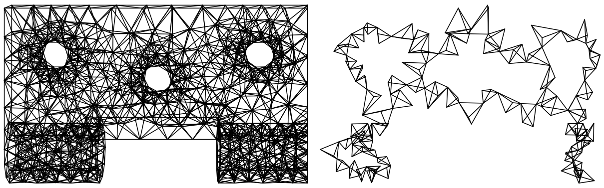
3.1.3. *Estimates for* $m=3$. We mostly analyzed the three dimensional case. Checking whether a simplex is flesh is $O(4+6+4+8) = O(22)$ because we need to count the relevant 2, 1 and 0-boundary and check the criteria. By chosing a subtle order in which we search for flesh after every deletion (only a former neighbour of a deleted simplex can become flesh), the search for flesh usually only takes $O(1)$ (if we can proceed with a neighbour of a deleted simplex) and a few times $O(n_i)$ (when every neighbor of the deleted simplex is a bone), where n_i is the cardinality of S_2 after the i-th deletion. This yields a complexity of $O\big(22\cdot((n_0\cdot k)+n_0-k)\big) = O\big(22\cdot(n_0\cdot(k+1)-k\big)$ where k is the number of "deep" searches for flesh. Giving estimates for k is difficult. But we can state that if k is large, say in the whereabouts of n_0, we know that almost every simplex initially was flesh and there was no need to be cautious in the first place when searching flesh.
We observed an overall complexity of about $O(n^{1.1})$.

3.2. **Gauging.** We analyzed objects of different topological properties and complexity in three dimensions:

A. a cube with a torus cut out
B. a hollow ball with three holes
C. two cubes, one of them with two enlinked tori cut out
D. a disk with 100 holes
E. seven balls having pairwise non-empty intersections
F. a solid "lego" brick
G. a door hinge

object	in	out	$\mathcal{D}$/ms	ms	$(\beta_0,\beta_1,\beta_2)$
A	1348	231	15	70	(1,1,1)
A	72328	5128	12	5347	(1,1,1)
A	90160	5705	9	8943	(1,1,1)
B	5524	68	21	251	(1,2,0)
B	64700	154	15	4126	(1,2,0)
C	4140	656	11	300	(2,2,2)
C	208624	12736	8	22953	(2,2,2)
D	30158	669	21	1392	(1,100,0)
D	164256	1325	16	10024	(1,100,0)
E	87610	1	14	6129	(1,0,0)
F	143192	1	16	8772	(1,0,0)
G	113352	304	15	7181	(1,5,0)

Where *in* is the initial number of 3-simplices, *out* the resulting number, $\mathcal{D}/\mathrm{ms}$ the number of cell deletions per millisecond, *ms* the overall simplification time in milliseconds and β_i is the i^{th} Betti number.



3.3. **Memory Usage.** During the runtime of the algorithm we often need to refer to the subsimplices of a simplex and also need to know which higher dimensional simplices a simplex belongs to. A data structure that allows us to access neighborhoods in both higher and lower dimensions will therefore be adequate to improve efficiency. In our test algorithm we implemented a data structure *simplex*:

```
class simplex {
  int dimension;
  simplex[][] IncidesWith;
  boolean isDeleted;}
```

where $IncidesWith$ is an array of simplex arrays. Example: $IncidesWith[k][l]$ is the l^{th} subsimplex of dimension k, if $k < dimension$, and it is the l^{th} higher dimensional simplex of which `simplex` is a face, if $k > dimension$.
Usually the memory needed to store a simplicial complex in this way grows linearly with the size of the complex. The fact that we store pointers to subsimplices and parental-simplices makes it difficult to predict memory usage generally: If the complex was generated as a Delaunay mesh (or almost as one), the memory defacto scales linearly; otherwise one cannot predict anything.
We observed that the memory usage during runtime remains constant.

3.4. **Results.** We observed in visualizations, that quasi-minimal complexes are easier to comprehend (concerning number of holes and similar) than the original complexes. The walls of a hollow ball will be exactly one simplex thick and a triangulated ring becomes a chain of single attached simplices.
To compute the Betti numbers β_i we used a slight modification of the algorithm described in [3]. We observed, that β_i remained the same ($\forall i$) for all valid cell deletions and that any cell deletion in a quasi-minimal subcomplex changed at least one β_j. Cell deletions can be expressed in terms of row- and column-deletions in the incidence matrices.

Remark. In the introduction we stated that the number of neighbors of a simplex remains the same (or decreases) during cell deletions but is likely to increase during edge contractions. If for example an edge is part of many triangles, then there are more non-zero entries in the corresponding row in the incidence matrix. The decomposition will therefore be more costy. It is subject to ascertain which methods weaknesses carries more weight.

Appendix A. Notes on Cell Deletions

A.1. **Further Simplifications.** The simplicial complex returned by our algorithm is a quasi-minimal subcomplex of the *same* homogeneous dimension as the original one. It is possible, when we omit this constraint, to further reduce the number of simplices used. Given the simplified complex, we can "deflate" certain simplices without modifying the Betti numbers.

Remark (Example)**.** A triangulated ball will be reduced to a single simplex of maximal dimension (and its boundary). A deflation would result in a single vertex.

Furthermore, as the result is only quasi-minimal, we can try to improve our result by certain local modifications, e.g. we could remove a simplex of maximal dimension and attach its former neighbours each to each other using edge contractions or just by exchanging vertices. The drawback however is, that we do not deal with a subcomplex anymore.

A.2. **Perfect Simplification?** We claimed, that $\mathfrak{M}$ will be simplified to a single simplex (and its boundary), if $|\mathfrak{M}|$ is homotopy equivalent to one of its vertices. This is equivalent to saying

$$|\mathfrak{M}| \approx p \in \mathbb{R}^m \implies \exists \tau \in \mathfrak{M} : \tau \text{ is flesh}$$

As the criteria for higher dimensional flesh depend on this claim a deeper insight on its validity would be apropriate; however we did not succeed to prove nor disprove it. Fortunately our experiments have not shown any counterexamples.

Appendix B. Outlook

We combined our results with the algorithm described in [3] to compute the betti numbers. It would be interesting to compute generators of the homology groups of the *simplified* simplicial complex $\mathfrak{Q}$ and then subsequently track them back in $\mathfrak{Q} \uplus \eta$, $(\mathfrak{Q} \uplus \eta) \uplus \nu$, $\big((\mathfrak{Q} \uplus \eta) \uplus \nu\big) \uplus \tau$ etc. and analyze the complexity of the trackback algorithm.
Combination with other simplification methods like described in [1] or [2] could improve the simplification of our algorithm and finally give optimal estimates for the needed number of simplices for the triangulation of a manifold of genus g.

References

[1] Olivier Devillers. On deletion in delaunay triangulation. *Internat. J. Comput. Geom. Appl., 12:193-205*, 2002.

[2] T. Dey, H. Edelsbrunner, S. Guha, and D. Nekhayev. Topology preserving edge contraction, 1998.

[3] F. Rapetti et al. Integer matrix factorization for mesh defect detection. *C. R. Acad. Sci. Paris*, pages 717–720, 2002.

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