Sliding Interfaces for Eddy Current Simulations

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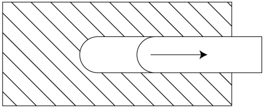
Outline

Motivation

- Generator circuit breakers
  - translational motion
- Electric engines
  - rotation



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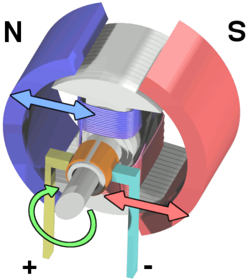

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Maxwell's Equations

$$\operatorname{div} \mathbf{B} = 0 \qquad \mathbf{curl}\,\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \mathbf{curl}\,\mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0(\mathbf{j}^{\mathbf{f}} + \mathbf{j}^{\mathbf{i}}).$$

$\Downarrow\ \mathbf{j}^f = \sigma \mathbf{E}, c \to \infty$ { Quasistatic model for slowly varying Electric fields (High conductivities)

Eddy Current Model

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$$\mathbf{curl}\, \mathbf{B} = \mu_0 \left(\sigma \mathbf{E} + \mathbf{j}^i\right) \qquad \operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Maxwells equations are invariant under Lorentz transformation if $\mathbf{E}$ and $\mathbf{B}$ transform as

$$\tilde{\mathbf{E}} = \gamma(\mathbf{E} + \mathbf{V} \times \mathbf{B}) - (\gamma - 1)(\mathbf{E} \cdot \hat{\mathbf{V}})\hat{\mathbf{V}}$$

$$\tilde{\mathbf{B}} = \gamma \left(\mathbf{B} - \frac{\mathbf{V} \times \mathbf{E}}{c^2} \right) - (\gamma - 1)(\mathbf{B} \cdot \hat{\mathbf{V}})\hat{\mathbf{V}}$$

$$\gamma := \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \hat{\mathbf{V}} = \hat{\mathsf{v}}/|\hat{\mathsf{v}}|$$

$$\Downarrow c \to \infty$$

$$\tilde{\mathbf{E}} = \mathbf{E} + \mathbf{V} \times \mathbf{B}$$

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It can be shown that the eddy current model is also invariant under *Rotation*!!!

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Two eddy current formulations

Temporal gauged Potential formulation :

$$\begin{aligned} \mathbf{curl}\,\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} + \sigma\frac{\partial \mathbf{A}}{\partial t} &= \mathbf{j}^i \\ \mathbf{A}(t=0) &= 0 \\ \mathbf{curl}\,\mathbf{A} \times \mathbf{n} &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

$\mathcal{H}$-formulation :

$$\begin{aligned} \mathbf{curl}\,\frac{1}{\sigma}\,\mathbf{curl}\,\mathcal{H} + \mu\frac{\partial \mathcal{H}}{\partial t} &= \mathbf{curl}\,\frac{1}{\sigma}\mathbf{j}^i \\ \mathcal{H}(t=0) &= 0 \\ \mathcal{H} &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Two eddy current formulations

Temporal gauged Potential formulation (Rest frame):

$$\mathbf{curl}\,\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} + \sigma\frac{\partial \mathbf{A}}{\partial t} = \mathbf{j}^i + \sigma \mathbf{V} \times \mathbf{curl}\,\mathbf{A}$$

$$\mathbf{A}(t=0) = 0$$

$$\mathbf{curl}\,\mathbf{A} \times \mathbf{n} = 0 \qquad \text{on } \partial\Omega$$

$\mathcal{H}$-formulation (Rest frame):

$$\mathbf{curl}\,\frac{1}{\sigma}\,\mathbf{curl}\,\mathcal{H} + \mu\frac{\partial \mathcal{H}}{\partial t} = \mathbf{curl}\,\frac{1}{\sigma}\mathbf{j}^i + \mathbf{curl}\,(\mu \mathbf{V} \times \mathcal{H})$$

$$\mathcal{H}(t=0) = 0$$

$$\mathcal{H} = 0 \qquad \text{on } \partial\Omega$$

Two eddy current formulations

Temporal gauged Potential formulation (Moving frame) :

$$\tilde{\mathbf{curl}}\frac{1}{\mu}\tilde{\mathbf{curl}}\tilde{\mathbf{A}} + \sigma\frac{\partial\tilde{\mathbf{A}}}{\partial t} = \tilde{\mathbf{j}}^i$$
$$\tilde{\mathbf{A}}(t=0) = 0$$
$$\tilde{\mathbf{curl}}\tilde{\mathbf{A}} \times \mathbf{n} = 0 \qquad \text{on } \partial\Omega$$

$\mathcal{H}$-formulation (Moving frame):

$$\tilde{\mathbf{curl}}\frac{1}{\sigma}\tilde{\mathbf{curl}}\tilde{\mathcal{H}} + \mu\frac{\partial\tilde{\mathcal{H}}}{\partial t} = \tilde{\mathbf{curl}}\frac{1}{\sigma}\tilde{\mathbf{j}}^i$$
$$\tilde{\mathcal{H}}(t=0) = 0$$
$$\tilde{\mathcal{H}} = 0 \qquad \text{on } \partial\Omega$$

Note: If $\mathbf{j}^i$ is smooth enough, $\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} = \mathcal{H}$

$\Rightarrow$ Do the same simulation and compare the two models (Primal & Dual formulation).

Transformation laws

The coordinates of the moving frame ($\tilde{\mathbf{x}}$) are related to the rest frame ($\mathbf{x}$) by

$$\mathbf{x} = \mathbf{T}(t)\tilde{\mathbf{x}} + \mathbf{r}(t).$$

T: Rotation matrix.

Transformation laws

$$\mathbf{T}\tilde{\mathbf{E}} = \mathbf{E} + \mathrm{V} \times \mathbf{B} \qquad \mathbf{T}\tilde{\mathbf{B}} = \mathbf{B}$$

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$$\mathbf{T}\tilde{\mathbf{j}^{\mathbf{f}}} = \mathbf{j}^{\mathbf{f}} \qquad \mathbf{T}\tilde{\mathbf{V}} = -\mathbf{V}$$

$$\mathbf{T}\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{T}\int_0^t \mathbf{T}^T \,\mathbf{grad}\,(\mathbf{V} \cdot \mathbf{A})$$

$\Rightarrow$ Use transformation laws to derive transmission conditions at sliding interface.

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Outline

DG Formulation of the Eddy Current Model

$$\sigma \frac{\partial \mathbf{A}}{\partial t} + \mathbf{curl}\, \frac{1}{\mu}\, \mathbf{curl}\, \mathbf{A} = \mathbf{j}^i$$

$$\mathbf{curl}\, \mathbf{A} \times \mathbf{n} = 0 \qquad \text{on } \partial\Omega$$

DG Variational formulation

Find $\mathbf{A}_h^{(i)} \in V_h$, $i = 1, \ldots, N$ such that for all $\mathbf{A}'_h \in V_h$, we have

$$\left(\sigma \frac{\mathbf{A}_h^{(i+1)} - \mathbf{A}_h^{(i)}}{\delta t}, \mathbf{A}'_h\right) + a_h^{\text{SWIP}}(\mathbf{A}_h^{(i+1)}, \mathbf{A}'_h) = \left(\mathbf{j}^{i,(i+1)}, \mathbf{A}'_h\right)$$

Where $V_h := \left[\mathbb{P}_3^k(\mathcal{T}_h)\right]^3$, $\mathbb{P}_d^k(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) \middle|\ \forall T \in \mathcal{T}_h,\ v|_t \in \mathbb{P}_d^k(T) \right\}$.

Symmetric-Weighted-Interior-Penalty Bilinear form a_h^{SWIP}

$$
\begin{aligned}
a_h^{\text{SWIP}}(\mathbf{A}_h, \mathbf{A}'_h) = {} & \int_\Omega \frac{1}{\mu} \mathbf{curl}_h \mathbf{A}_h \cdot \mathbf{curl}_h \mathbf{A}'_h \\
& - \sum_{F \in \mathcal{F}_h^i} \int_F \left\{ \frac{1}{\mu} \mathbf{curl}_h \mathbf{A}_h \right\}_\omega \cdot \left[\mathbf{A}'_h\right]_T \\
& - \sum_{F \in \mathcal{F}_h^i} \int_F \left\{ \frac{1}{\mu} \mathbf{curl}_h \mathbf{A}'_h \right\}_\omega \cdot \left[\mathbf{A}_h\right]_T \\
& + \sum_{F \in \mathcal{F}_h^i} \frac{\eta \gamma_{\mu,F}}{h_F} \int_F \left[\mathbf{A}_h\right]_T \cdot \left[\mathbf{A}'_h\right]_T
\end{aligned}
$$

$$
\{\mathbf{A}_h\}_\omega = \omega_1 \mathbf{A}_{h,1} + \omega_2 \mathbf{A}_{h,2}, \qquad [\mathbf{A}_h]_T = \mathbf{n}_F \times (\mathbf{A}_{h,1} - \mathbf{A}_{h,2}) \tag{1}
$$

$$
\omega_1 = \frac{\mu_1}{\mu_1 + \mu_2}, \quad \omega_2 = \frac{\mu_2}{\mu_1 + \mu_2}, \quad \gamma_{\mu,F} = \frac{2}{\mu_1 + \mu_2} \tag{2}
$$

Convergence

Under regularity conditions on the mesh sequence (matching) and assuming the exact solution $\mathbf{A}$ is smooth enough we can prove

$$\left\|\sqrt{\sigma}(\mathbf{A}^{(N)} - \mathbf{A}_h^{(N)})\right\|_{L^2(\Omega)} + \left(C_{\text{stab}}\delta t \sum_{i=1}^{N} \left|\mathbf{A}^{(i)} - \mathbf{A}_h^{(i)}\right|^2_{\text{SWIP}}\right)^{1/2} \leq Ct_F^{1/2}\left(C_1 h^k + C_2 \delta t\right)$$

where $C_1 = \max_{t\in[0,t_F]} |\mathbf{A}(t)|_{H^{k+1}(\Omega)}$ and $C_2 = \max_{t\in[0,t_F]} \left\|\frac{\partial^2 \mathbf{A}(t)}{\partial t^2}\right\|_{L^2(\Omega)}$ The constants C_1,C_2 and C are independent of h and δt.

$$|\mathbf{A}|_{\text{SWIP}} := \left(\left\|\frac{1}{\sqrt{\mu}}\,\mathbf{curl}_h\,\mathbf{A}\right\|^2_{L^2(\Omega)} + \sum_{F\in\mathcal{F}_h} \frac{\gamma_{mu,F}}{h_F}\,\|[\mathbf{A}]_T\|^2_{L^2(F)}\right)^{1/2}$$

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Under regularity conditions on the mesh sequence (matching) and assuming the exact solution $\mathbf{A}$ is smooth enough we can prove

$$\left\| \sqrt{\sigma}(\mathbf{A}^{(N)} - \mathbf{A}_h^{(N)}) \right\|_{L^2(\Omega)} + \left(C_{\text{stab}} \delta t \sum_{i=1}^{N} \left| \mathbf{A}^{(i)} - \mathbf{A}_h^{(i)} \right|_{\text{SWIP}}^2 \right)^{1/2} \leq C t_F^{1/2} \left(C_1 h^k + C_2 \delta t \right)$$

where $C_1 = \max_{t \in [0,t_F]} |\mathbf{A}(t)|_{H^{k+1}(\Omega)}$ and $C_2 = \max_{t \in [0,t_F]} \left\| \frac{\partial^2 \mathbf{A}(t)}{\partial t^2} \right\|_{L^2(\Omega)}$ The constants C_1,C_2 and C are independent of h and δt.

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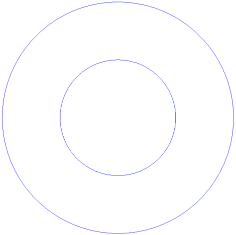
Aspects of the implementation

- In comparison to FEM, DG has much more degrees of freedom
  - Use DG only along the non-matching interfaces.
- Incorporate the transformation formulas for the moving frame into the DG fluxes.
  - No convective terms appear
- 2D spatial discretization:
  - 1st order Edge functions of the first kind for *vectorial problem*.
  - 1st order Lagrange elements for *scalar problem*.
- Use NGSolve and Netgen.

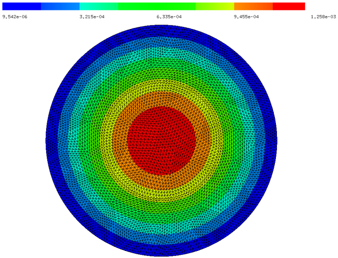
Outline

Convergence analysis with analytical solution

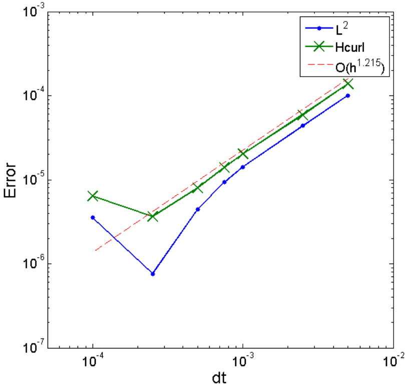
- Construct an analytical radial solution, $\mathcal{H}_z = \mathcal{H}_z(|\mathbf{x}|)$ to 2D scalar $\mathcal{H}$-formulation (TE).
- Let Ω_1 rotate at $\omega = 20\text{rad}/\text{s}$.
- Measure rate of convergence in L^2 and $SWIP$-norm.



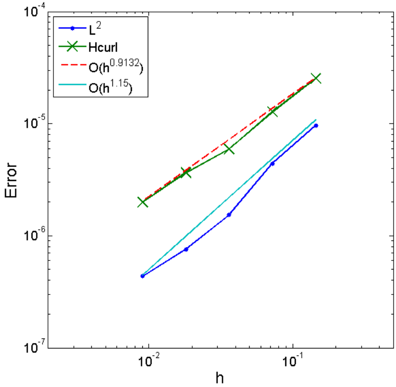
(a) The subdomains Ω_1 and Ω_2



(b) The solution at time $t = 0.05$ sec($\delta t = 0.0005$, $h = 0.0361371$)



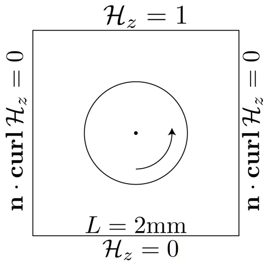
(a) δt convergence, $h = 0.0180874$

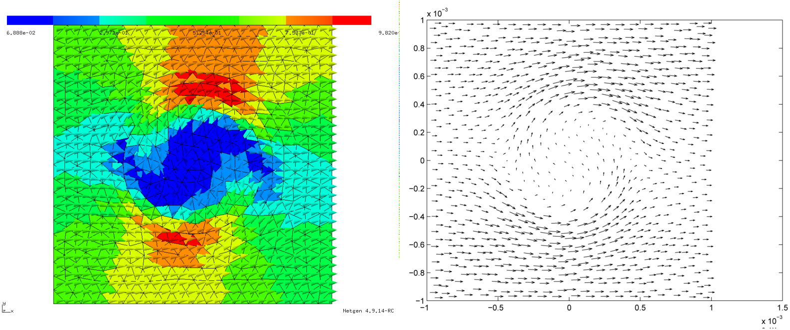


(b) h convergence, $\delta t = 2.5e - 4$

Comparison with F. Rapetti et al.

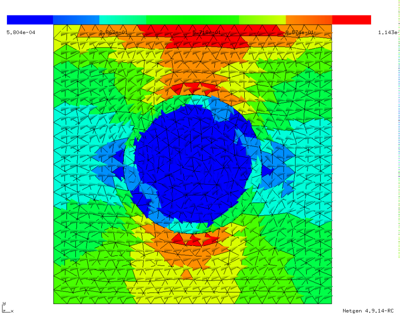
- F. Rapetti used Mortar Method to deal with non-conforming mesh.
- 2D, scalar $\mathcal{H}$-formulation (Transverse magnetic).
- Simulation time: 0.2s.
- $\mathbf{j}^i = 0$





(a) DG, background shows $|\mathbf{curl}_{2D}\,\mathcal{H}_z|$ (b) Mortar method

Figure: Visualization of $\mathbf{curl}_{2D}\,\mathcal{H}_z$ for $\omega = 630\text{rad}/\text{s}$



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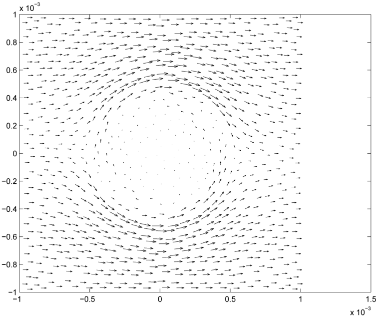

(b) Mortar method

Figure: Visualization of $\mathbf{curl}_{2D}\,\mathcal{H}_z$ for $\omega = 6300\text{rad}/\text{s}$

Complex rotational setting

- Circle rotating in square (as before).
- Compare $\mathcal{H}$-formulation with temporal gauged $\mathbf{A}$-formulation by measuring $\left\| \frac{1}{\mu} \operatorname{curl}_{2D} \mathbf{A} - \mathcal{H}_z \right\|_{L^2(\Omega)}$.
  - $\mathbf{A}$: 2D vector
  - $\mathcal{H}$: 1D scalar
- Excitation by impressed current $\mathbf{j}^i = (4y, -2x)$.
- $\omega = 4\pi$, $t_{\text{end}} = 1$.
- $\mathbf{T}\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{T} \int_0^t \mathbf{T}^T \, \mathbf{grad} \, (\mathbf{V} \cdot \mathbf{A})$

(Movies)

Convergence

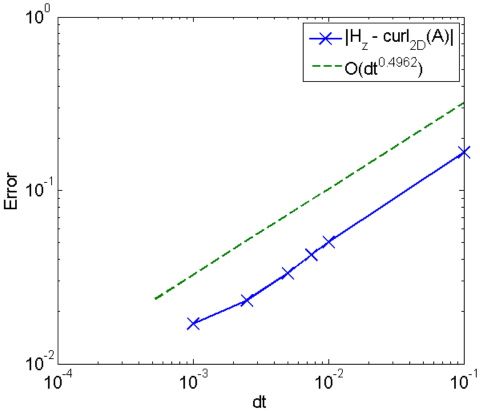

Figure: δt convergence at $t_{\text{end}} = 1$ (one full rotation), $h = 0.0254402$.

Conclusion

- DG approach is a viable alternative to Mortar methods for simulating sliding interfaces.
- The $\mathcal{H}$-formulation is equivalent to the temporal gauged potential formulation if the correct transformation rules are used.
- $O(h)$ and $O(t)$ convergence was proven for a system at rest.
- Outlook
  - Coulomb gauged potential formulation: No time integration is needed
  - Extension to 3D

Questions ?