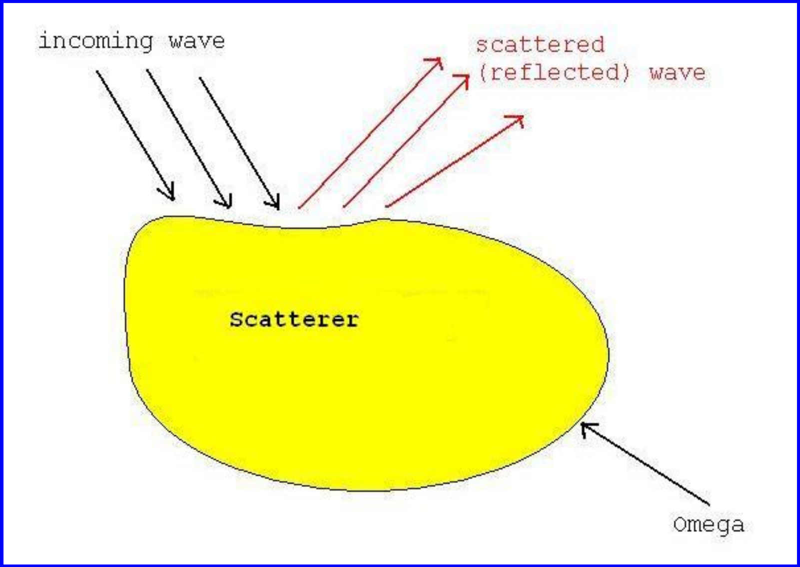
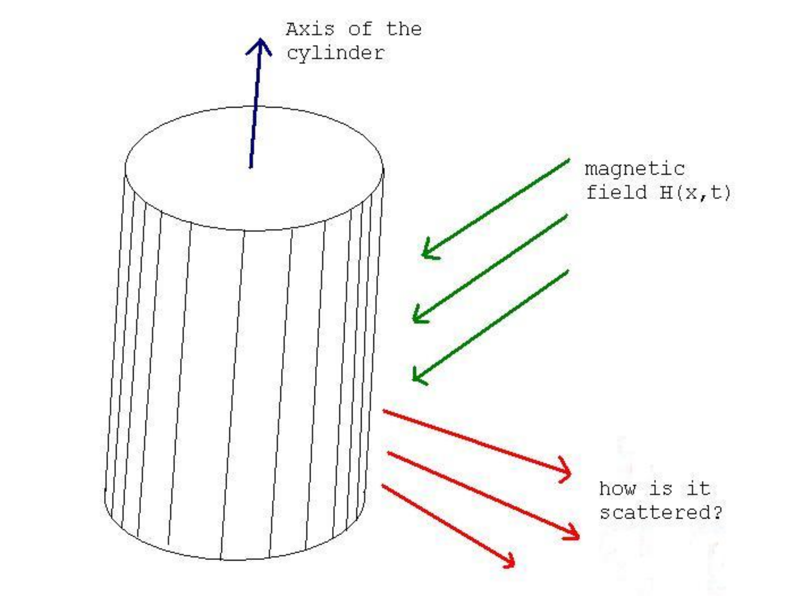
Analysis of PML for the Helmholtz equation

What's the physical situation?

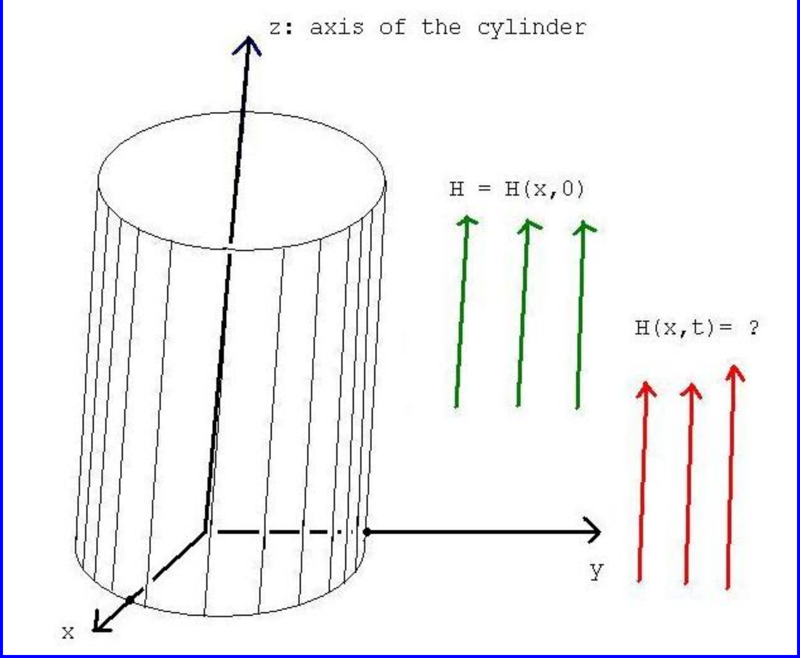


Example

Let Ω represent an infinite cylinder in $\mathbb{R}^3$. We consider how the magnetic field $\mathbf{H} = \mathbf{H}(\mathbf{x}, t)$ is scattered by the cylinder.



We assume that the magnetic field is parallel to the axis of the cylinder.



With further assumptions, the scattered magnetic field can be written as

$$\mathbf{H} = u(r, \phi)e^{-iwt}\mathbf{e}_z$$

- where (r, ϕ) are the polar coordinates in the plane,
- $w > 0$ the angular frequency,
- $\mathbf{e}_z$ the direction of the axis of the cylinder.

Mathematical analysis of the situation

- Mathematical description?
- How to describe „the scattering“?

Answer (w.r.t. the Example)

For the amplitude $u = u(r, \phi)$ we have to solve:

1. $(\Delta + k^2)u = 0$ in $\mathbb{R}^2 \setminus \bar{\Omega}$,
2. $\frac{\partial u}{\partial n}|_{\partial\Omega} = g$,
3. $\lim_{r\to\infty} r^{1/2}(\frac{\partial u}{\partial r} - iku) = 0$ uniformly in $\hat{x}$,

where $k^2 = w^2\epsilon_0\mu_0 > 0$ and

$r = |x|$ and $\hat{x} = x/r, \quad r \neq 0$.

the equation (3) is called the
„Sommerfeld radiation condition at infinity“.

Difficulties

Since Ω is bounded, $\mathbb{R}^2 \backslash \bar{\Omega}$ is unbounded.

Solution: The PML - method.

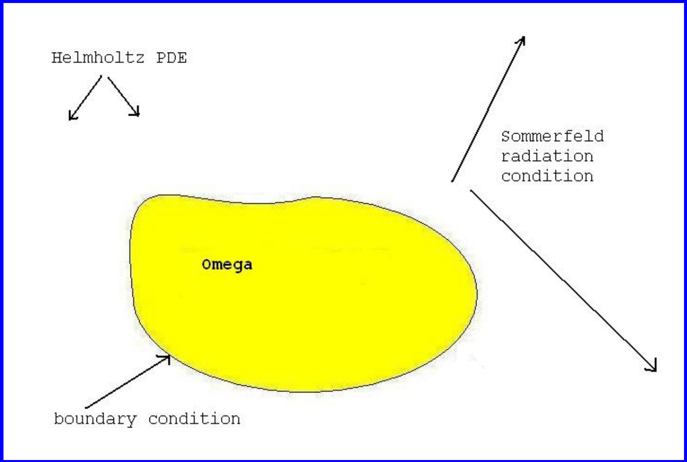
Overview

1. Scattering BVP $\longrightarrow u_{sc}$.
2. Full-space Bérenger BVP $\longrightarrow u_{\mathbb{C}}$ with $u_{\mathbb{C}}\big|_D = u_{sc}\big|_D$.
3. Truncated Bérenger BVP $\longrightarrow \widetilde{u}_{\mathbb{C}}(\rho)$, bounded domain.
4. Main Theorem: $\widetilde{u}_B(\rho) \stackrel{\rho\to\infty}{\longrightarrow} u_{sc}$ near Ω.
5. Outline of the proof.

Part 1: The scattering BVP

1. $(\Delta + k^2)u = 0$ in $\mathbb{R}^2 \setminus \bar{\Omega}$
2. $\frac{\partial u}{\partial n}|_{\partial\Omega} = g \in H^{-1/2}(\partial\Omega)$
3. $\lim_{r\to\infty} r^{1/2}(\frac{\partial u}{\partial r} - iku) = 0$ uniformly in $\hat{x}$

Note: $(3) \Longrightarrow u(r,\varphi) \overset{r\to\infty}{\longrightarrow} C_1\, e^{ikr+C_2\varphi}$



Existence Theorem

The scattering BVP has a unique (weak) solution $u \in H^1_{rad}(\mathbb{R}^2 \backslash \bar{\Omega})$, where

$$H^1_{rad}(\mathbb{R}^2 \backslash \bar{\Omega}) = \{u \; ; \; u \in H^1(B_R(0) \backslash \bar{\Omega}) \text{ for all } R > 0$$

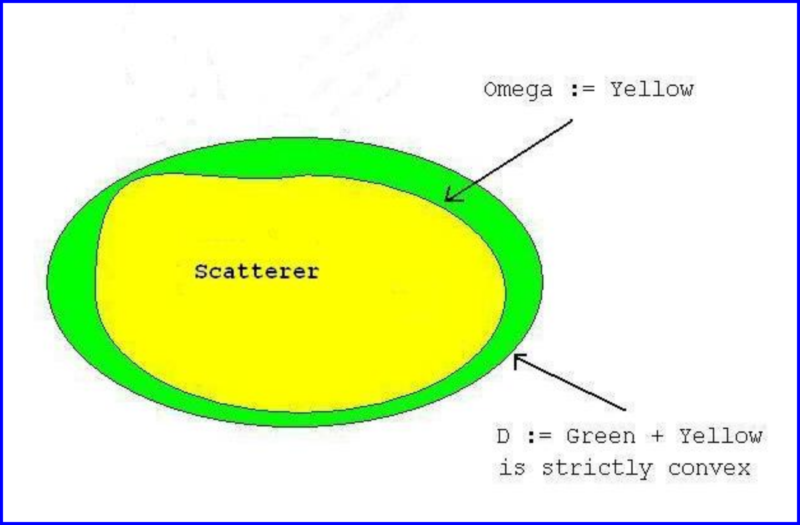
$$\text{and } u \text{ satisfies the Sommerfeld radiation condition } \}.$$

This unique solution of the scattering problem is denoted by u_{sc}.

Part 2: The Construction

We introduce a strictly convex set $D \subset \mathbb{R}^2$, such that

1. $\bar{\Omega} \subset D$,
2. D has a C^2-boundary.

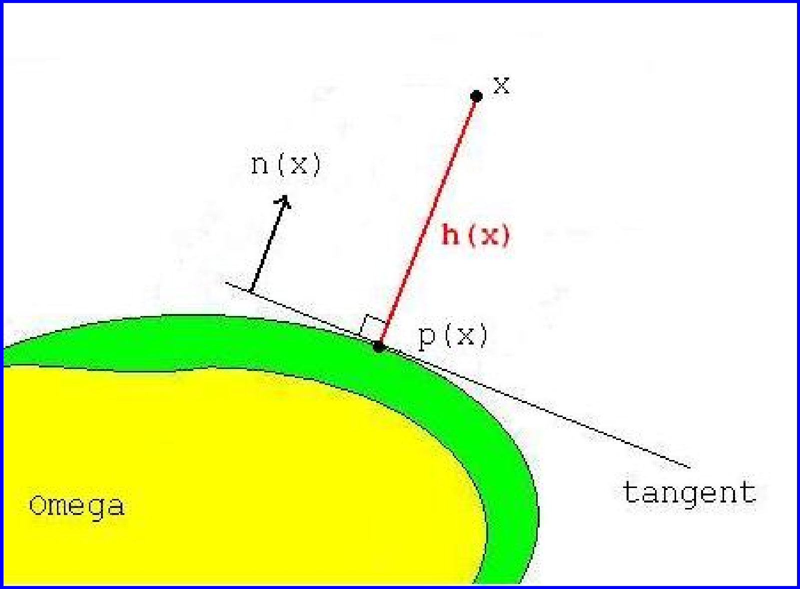


Definition of h

Let $x \in \mathbb{R}^2 \backslash D$. Then we define

- $h(x) := dist(x, \partial D) \geq 0$ and
- $p(x) \in \partial D$ such that $h(x) = |x - p(x)|$.

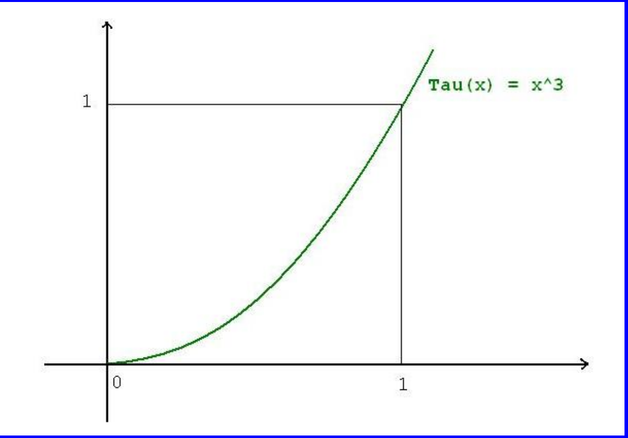
With $x = p(x) + h(x) \cdot \mathbf{n}(x)$.



Definition of τ

Further, let $\tau : [0, \infty) \longrightarrow [0, \infty)$ be a C^2-function with:

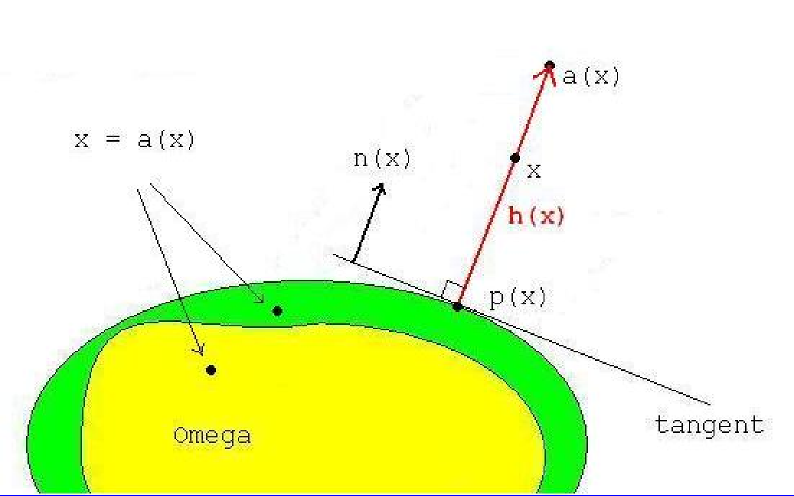
- the derivative τ' is strictly increasing,
- $\lim_{s\to\infty} \tau'(s) = \infty$
- $\tau(0) = \tau'(0+) = \tau''(0+) = 0$
- $\lim_{s\to\infty} e^{-\epsilon\cdot\tau(s)}\tau'(s) = \lim_{s\to\infty} e^{-\epsilon\cdot\tau(s)}\tau''(s) = 0$ for all $\epsilon > 0$.



Definition of the function a

We define a function $a : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by setting

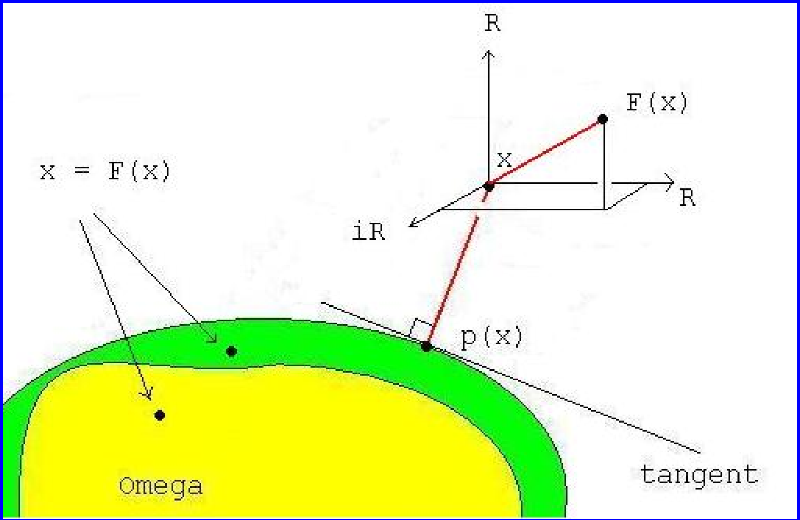
$$a(x) := \begin{cases} 0 & \text{if } x \in D \\ \tau(h(x)) \cdot \mathbf{n}(x) & \text{if } x \in \mathbb{R}^2 \setminus \overline{D} \end{cases}$$



Definition of the stretching function F

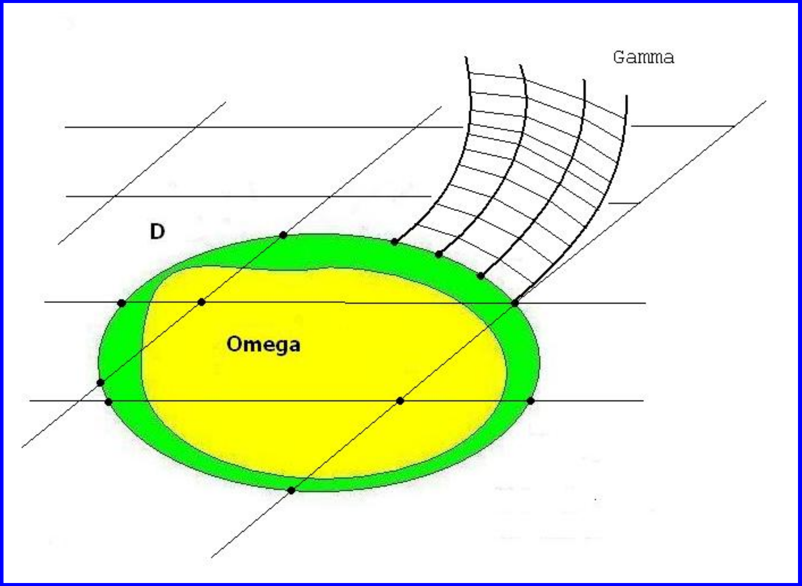
We define the function $F : \mathbb{R}^2 \longrightarrow \mathbb{C}^2$ by setting

$$F(x) := x + i \cdot a(x).$$



Definition of Γ

$$\Gamma := F(\mathbb{R}^2) = \{z \in \mathbb{C}^2 \; ; \; z = x + i \cdot a(x), x \in \mathbb{R}^2\}$$



The fundamental solution of the Helmholtz equation

$$\Phi(x, y) = \frac{i}{4} \cdot H_0^{(1)}(k \cdot |x - y|)$$

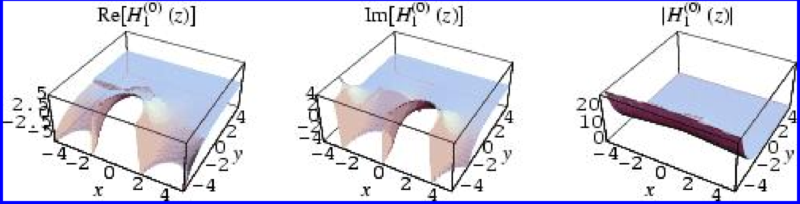
is the **fundamental solution** for the Helmholtz equation, i.e.

$$(\Delta_y + k^2)\Phi(x, y) = \delta(x).$$

The **Hankel function** of the first kind:

$$H_0^{(1)} : \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto \frac{1}{i\pi} \int_0^\infty \frac{e^{(z/2)(t-1/t)}}{t}\, dt$$

with $H_0^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{i(x-\pi/4)}$ for $x \in \mathbb{R}$ and $x \gg 1$.



Representation of u_{sc}

Definition:

- $S_{\partial\Omega,\mathbb{R}^n\setminus\overline{\Omega}}\left[\varphi\right](x) := \int_{\partial\Omega}\Phi(x,y)\varphi(y)dS(y)$

  - Single layer potential operator

- $K_{\partial\Omega,\mathbb{R}^n\setminus\overline{\Omega}}\left[\psi\right](x) := \int_{\partial\Omega}\frac{\partial\Phi(x,y)}{\partial n(y)}\psi(y)dS(y)$

  - Double layer potential operator

Representation:

$$u_{sc} = S_{\partial\Omega,\mathbb{R}^n\setminus\overline{\Omega}}\left[\varphi\right] + K_{\partial\Omega,\mathbb{R}^n\setminus\overline{\Omega}}\left[\psi\right]$$

with some densities φ and ψ.

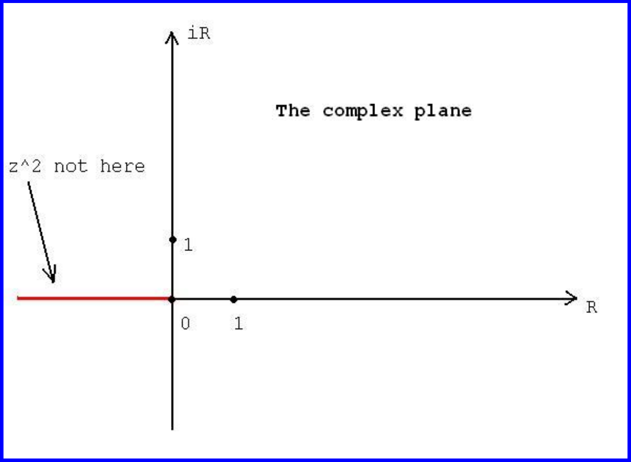
Complexification of the function $|\cdot|$

The function $\rho(x) := |x|$ for $x \in \mathbb{R}^2$ allows an analytic extension to $G \subset \mathbb{C}^2$, where

$$G := \{z = (z_1, z_2) \in \mathbb{C}^2 \; ; \; z^2 = z_1^2 + z_2^2 \in \mathbb{C}^2 \backslash (-\infty, 0)\}$$

This extension is denoted by ρ again and we have:

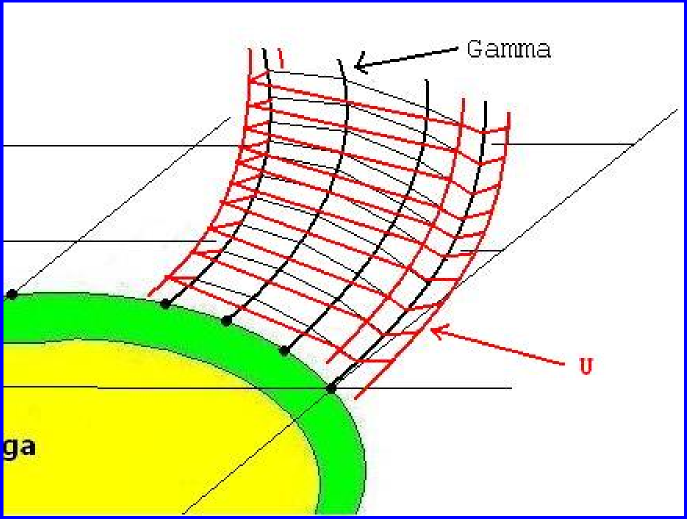
$$\rho : G \longrightarrow \{z \in \mathbb{C} \; ; \; \Re z > 0\}$$



Neighborhood of Γ

Lemma: The manifold $\Gamma \backslash \bar{\Omega} \subset \mathbb{C}^2$ has a neighborhood $U \subset \mathbb{C}^2$ such that, for all $y \in \partial\Omega$ and $z \in U$ we have $z - y \in G$, i.e.

$$(z_1 - y_1)^2 + (z_2 - y_2)^2 \notin \mathbb{R}_-$$



Extension of the fundamental solution Φ

Lemma: The Hankel function $H_0^{(1)}$ is analytic in $\{z \in \mathbb{C} \,|\, \Re z > 0\}$.

Definition:

$$\Phi(z,\zeta) := \frac{i}{4} H_0^{(1)}(k \cdot \rho(z-\zeta)), \quad z-\zeta \in G.$$

Note:

„Extended $\Phi \longrightarrow$ extended S and $K \longrightarrow$ extended u_{sc}“

Extensions

Analytic extensions of the Operators S and K:

- $S_{\partial\Omega,U}\left[\varphi\right](z) := \int_{\partial\Omega} \Phi(z,y)\varphi(y)dS(y), \quad z \in U$

- $K_{\partial\Omega,U}\left[\psi\right](z) := \int_{\partial\Omega} \frac{\partial\Phi(z,y)}{\partial n(y)}\psi(y)dS(y), \quad z \in U$

Definition: $u : U \longrightarrow \mathbb{R}$, with

$$u(z) := S_{\partial\Omega,U}\left[\varphi\right](z) + K_{\partial\Omega,U}\left[\psi\right](z)$$

Properties of the function $u = u(z)$

1. $z \longmapsto u(z)$ is $\mathbb{C}^2$-analytic in U.
2. $u(.)\Big|_{D\setminus\overline{\Omega}} = u_{sc}\Big|_{D\setminus\overline{\Omega}}$.
3. $z \longmapsto u(z)$ satisfies the complexified Helmholtz equation in U,

   $$(\Delta_z + k^2)\, u(z) = 0,$$

   where $\Delta_z = \partial_{z_1}^2 + \partial_{z_2}^2$.
4. $u \circ F(x) \overset{|x|\to\infty}{\longrightarrow} 0$ exponentially.

Representation of $\Delta_z u(z)$

Let u be an analytic function defined in a neighborhood of $\Gamma \subset \mathbb{C}^2$. Then, for $z \in \Gamma$,

$$\Delta_z u(z) = (\text{div } H^T H \text{ grad - } m^T H \text{ grad}) \left[u \circ F\right] (F^{-1}(z)),$$

where

- $H = (\underbrace{I + i(Da)}_{DF})^{-T} = \begin{pmatrix} 1 + i\frac{\partial a_1}{\partial x_1} & i\frac{\partial a_1}{\partial x_2} \\ i\frac{\partial a_2}{\partial x_1} & 1 + i\frac{\partial a_2}{\partial x_2} \end{pmatrix}^{-T}$
- $m = \begin{pmatrix} \frac{\partial}{\partial x_1}(H)_{1,1} + \frac{\partial}{\partial x_2}(H)_{1,2} \\ \frac{\partial}{\partial x_1}(H)_{2,1} + \frac{\partial}{\partial x_2}(H)_{2,2} \end{pmatrix}$

Corollary

The „Bérenger equation“ $[(\Delta_z + k^2) \cdot u(z)]\,|_\Gamma = 0$ assumes in $\mathbb{R}^2$ the form

$$(\text{div } H^T H \text{ grad - } m^T H \text{ grad} + k^2)\ [u \circ F] = 0.$$

Definition: $\widetilde{\Delta} := \text{div } H^T H \text{ grad - } m^T H \text{ grad}$

Complexified analogue of the space $H^1_{rad}(\mathbb{R}^2 \backslash \overline{\Omega})$

$H^1_{(\delta)}(\mathbb{R}^2 \backslash \overline{\Omega}) := \{u \in H^1(\mathbb{R}^2 \backslash \overline{\Omega})\ |$

$\lim_{h(x) \longrightarrow \infty} e^{\delta\tau(h(x))} \left| u(x) \right| = \lim_{h(x) \longrightarrow \infty} e^{\delta\tau(h(x))} \left| \text{grad} u(x) \right| = 0$

uniformly in $\widehat{x}\}$.

The full-space Bérenger problem

We want to find a function $u \in H^1_{(\delta)}(\mathbb{R}^2 \backslash \overline{\Omega})$ such that

1. $(\widetilde{\Delta} + k^2)u = 0$ in $\mathbb{R}^2 \backslash \overline{\Omega}$
2. $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g \in H^{-1/2}(\partial\Omega)$

Existence and uniqueness theorem

The full-space Bérenger problem has a unique solution $u_{\mathbb{C}} \in H^1_{k-\epsilon}(\mathbb{R}^2 \backslash \overline{\Omega})$, where $\epsilon > 0$ is arbitrary. Furthermore we have

$$u_{\mathbb{C}}\Big|_{D \backslash \overline{\Omega}} = u_{sc}\Big|_{D \backslash \overline{\Omega}}.$$

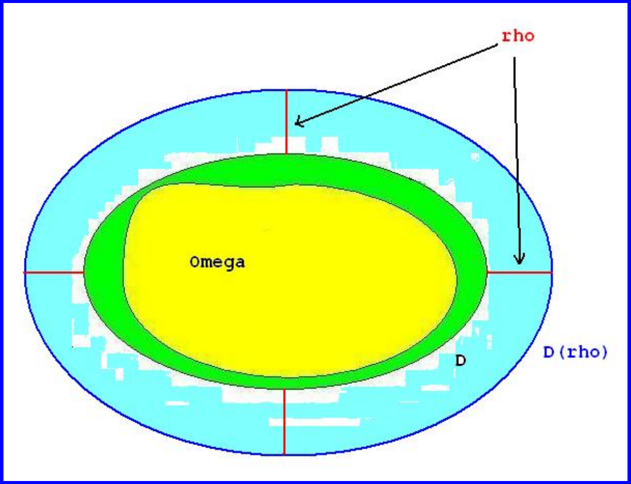
Part 3: Definition of the truncated Bérenger BVP

Definition: The layer of thickness $\rho > 0$ around D is defined by

$$L(\rho) := \left\{x \in \mathbb{R}^2 \backslash \overline{D} \mid h(x) < \rho\right\}.$$

We define further

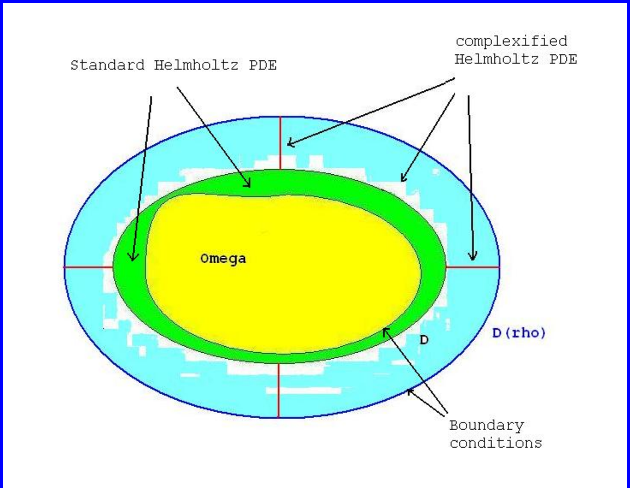
$$D(\rho) := D \cup L(\rho).$$



The truncated Bérenger problem

We want to find a function $u_T \in H^1(D(\rho)\backslash\overline{\Omega})$ satisfying

1. $(\widetilde{\Delta} + k^2)u = 0$ in $D(\rho)\backslash\overline{\Omega}$
2. $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g \in H^{-1/2}(\partial\Omega)$
3. $u\Big|_{\partial D(\rho)} = 0$



Part 4: Main theorem

For any wavenumber $k > 0$, there exists a positive constant $\rho_0(k)$ such that, for all $\rho \geq \rho_0(k)$, the truncated Bérenger problem (bounded) has a unique solution $u_T = u_T(\rho) \in H^1(D(\rho)\backslash\overline{\Omega})$.

Moreover, this solution converges exponentially to the solution u_{sc} of the initial scattering problem(unbounded) near Ω:

$\lim_{\rho\longrightarrow\infty} e^{(k-\epsilon)\tau(\rho)} \|u_{sc} - u_T(\rho)\|_{H^1(D(\rho)\backslash\overline{\Omega})} = 0$ for all $\epsilon > 0$.

It could be so easy...

By linearity of the operator $(\widetilde{\Delta}+k^2)$, we have for $\eta := u_{\mathbb{C}} - u_T$

1. $(\widetilde{\Delta}+k^2)\eta = 0$ in $D(\rho)\backslash\overline{\Omega}$
2. $\frac{\partial\eta}{\partial n}\,|_{\partial\Omega} = 0$
3. $\eta\,\Big|_{\partial D(\rho)} = u_{\mathbb{C}}$

If we could show that

$$\left\|\eta\right\|_{H^1(D\backslash\overline{\Omega})} \leq C\left\|u_{\mathbb{C}}\right\|_{H^{1/2}(\partial D(\rho))}, \quad C \text{ indep. of } \rho,$$

the main theorem was proved, since

$$\left\|u_{\mathbb{C}}\right\|_{H^{1/2}(\partial D(\rho))} \longrightarrow 0 \text{ as } \rho \longrightarrow \infty.$$

... but it isn't. $\rightarrow$

Part 5: Outline of the proof of the Main theorem

Three steps:

1. Full-space Bérenger BVP $\Longleftrightarrow$ BVP (A) near Ω
2. Truncated Bérenger BVP $\Longleftrightarrow$ BVP (B) near Ω
3. BVP (B) $\longrightarrow$ BVP (A) if the layer thickness $\rho \longrightarrow \infty$ near Ω.

Since we know that the full-space Bérenger BVP $\Longleftrightarrow$ scattering BVP near Ω, the Main Theorem is then proved.

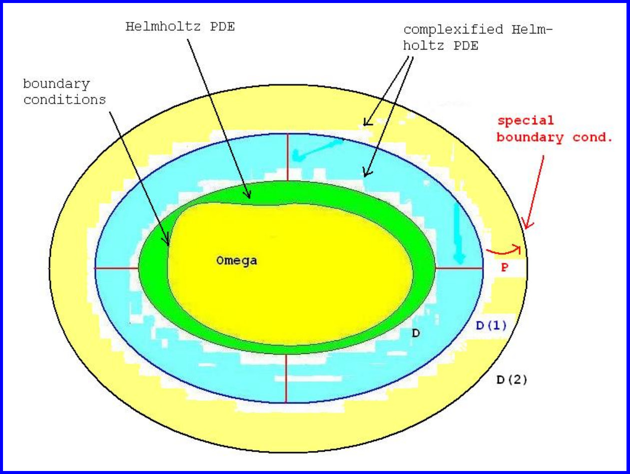
The idea behind step 1

Let $0 < \rho_1 < \rho_2$ and $D_j := D(\rho_j), \quad j = 1, 2.$
Find u with $(\Delta + k^2)u = 0$ in $D_2 \backslash \Omega$ and

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g, \qquad u\,\Big|_{\partial D_2} = P(u\,\Big|_{\partial D_1})\ ,$$

with the double surface operator $P = K_{\partial D_1, \partial D_2}(\frac{1}{2} + K_{\partial D_1})^{-1}$,

$$K_{\partial D_1}\varphi(x) = \text{p.v.}\int_{\partial D_1} \frac{\partial \Phi}{\partial n(y)}(x, y)dS(y), \quad x \in \partial D_1.$$

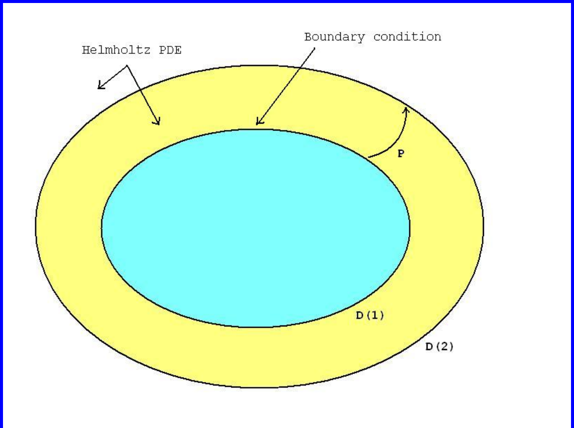


Characterization of P

If for a function u we have

1. $(\Delta + k^2)u = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D_1}$
2. $u\,|_{\partial D_1} = w,$

$$\Longrightarrow \quad Pw = u\,|_{\partial D_2}.$$



The Theorem behind step 1

Assume that ρ_1 and ρ_2 are so chosen that k^2 is not the Dirichlet - eigenvalue of $-\Delta$ in $D_2 \backslash \overline{D}_1$.

The BVP $(\Delta + k^2)u = 0$ in $D_2 \backslash \Omega$ with

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g, \qquad u\Big|_{\partial D_2} = P(u\Big|_{\partial D_1}) \, ,$$

has a unique solution u, and $u_{sc} \equiv u$ in D_2.

The task of step 1

- Find $P_{\mathbb{C}}$ analogous to P for the full-space Bérenger problem.
- Prove the „Theorem behind step 1“ with P replaced by $P_{\mathbb{C}}$.

Definition of the BVP (A)

Let the BVP (A) be defined by

1. $(\widetilde{\Delta} + k^2)u = 0$ in $D_2 \backslash \overline{\Omega}$
2. $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g \in H^{-1/2}(\partial\Omega)$
3. $u\,|_{\partial D_2} = P_{\mathbb{C}}(u\,|_{\partial D_1})$,

where $P_{\mathbb{C}} := \widetilde{K}_{(A),\partial D_1,\partial D_2}(\frac{1}{2} + \widetilde{K}_{(A),\partial D_1})^{-1}$,

$\widetilde{K}_{(A),\partial D_1,\partial D_2}[\psi](x) := \int_{\partial D_1} \frac{\partial \widetilde{\Phi}_{(A)}(x,y)}{\partial n(y)} \psi(y) dS(y)$,

$(\widetilde{\Delta} + A + k^2)\widetilde{\Phi}_{(A)}(x, y) = -\delta(x - y)$ and

$A = A(\epsilon) : L^2(D_1) \to L^2(D_1), \quad \| A \| < \epsilon$,

$\lim_{h(x)\to\infty} \sup_{y\in K\subset\mathbb{R}^2} e^{(k-\epsilon)\tau(h(x))} | D_x^\alpha \widetilde{\Phi}_{(A)}(x, y) | = 0, \quad |\alpha| \leq 2.$

The Theorem of step 1

The BVP (A) has a unique solution u in $H^1(D_2\backslash\overline{\Omega})$, and

$u = u_{\mathbb{C}}$ in $D_2\backslash\overline{\Omega}$.

Lemma

The BVP

1. $(\widetilde{\Delta} + k^2)u = 0$ in $\mathbb{R}^2\backslash\overline{D}_1$
2. $u\Big|_{\partial D_1} = f \;\in H^{1/2}(\partial D_1)$

has a unique solution $u \in H^1_{(1-\epsilon)}(\mathbb{R}^2\backslash\overline{D}_1)$ and it can be represented as $u = \widetilde{K}_{(A),\partial D_1,\mathbb{R}^2\backslash\overline{D}_1}[\varphi]$, where φ is the unique solution of

$(\frac{1}{2} + \widetilde{K}_{(A),\partial D_1})\,[\varphi] = f.$

The Theorem of step 2

Let $\rho > \rho_2$. There exists an operator

$$P_\rho : H^{1/2}(\partial D_1) \longrightarrow H^{1/2}(\partial D_2)$$

such that the truncated Bérenger problem is equivalent to the near-field BVP (B):

1. $(\widetilde{\Delta} + k^2)u = 0$ in $D(\rho)\backslash\overline{\Omega}$
2. $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g \in H^{1/2}(\partial\Omega)$
3. $u\Big|_{\partial D_2} = P_\rho(u\Big|_{\partial D_1})$.

Moreover, we have

$$\lim_{\rho\to\infty} e^{(k-\epsilon)\tau(\rho)} \| P_\rho - P_{\mathbb{C}} \| = 0 \text{ for all } \epsilon > 0.$$

Lemma

The BVP (C)

1. $(\widetilde{\Delta} + k^2)u = 0$ in $D(\rho)\backslash\overline{D}_1$
2. $u\Big|_{\partial D_1} = f \in H^{1/2}(\partial D_1)$
3. $u\Big|_{\partial D_\rho} = 0$

has a unique solution $u \in H^1(D(\rho)\backslash\overline{D}_1)$.

Step 3 - The connection between (A) and (B)

Assume that $\widetilde{P} : H^{1/2}(\partial D_1) \longrightarrow H^{1/2}(\partial D_2)$ is an operator with the property

$$\| \widetilde{P} - P_{\mathbb{C}} \| < \epsilon .$$

Consider the BVP (A) with $P_{\mathbb{C}}$ replaced by $\widetilde{P}$. For $\epsilon > 0$ small enough, that modified BVP has a unique solution $\widetilde{u} \in H^1(\partial D_2 \backslash \overline{\Omega})$, and we have

$$\left\| u_{\mathbb{C}} - \widetilde{u} \right\|_{H^1(D_2 \backslash \overline{\Omega})} < C\epsilon$$

for some positive constant $C > 0$.

Lemma

This BVP (D) is an „equivalent weak form of" the BVP (A)

1. $(\widetilde{\Delta} + k^2)u = Fu$ in $D_2 \backslash \overline{\Omega}$
2. $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g \in H^{-1/2}(\partial\Omega)$
3. $u\,|_{\partial D_2} = 0,$

where $Fu = -(\widetilde{\Delta} + k^2) R P_{\mathbb{C}}(u\,|_{\partial D_1})$ and

$R : H^{1/2}(\partial D_2) \to H^1(D_2 \backslash \overline{\Omega}),$

$R(u\,|_{\partial D_2}) = u$

a right inverse of the trace mapping $u \mapsto u\,|_{\partial D_2}$.

Thank you for your attention!