Construction of PML

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Contents

Introduction

Non-Maxwellian PML using complex coordinate stretching

$$x^i \longmapsto \tilde{x}^i = \int_0^{x^i} s_i(\hat{x}^i) d\hat{x}^i$$

with

$$s_i(\hat{x}^i) = a_i(\hat{x}^i) + i\sigma_i(\hat{x}^i)/\omega$$

$$\rightarrow \tilde{\nabla} = \begin{pmatrix} \partial/\partial\tilde{x}^1 \\ \partial/\partial\tilde{x}^2 \\ \partial/\partial\tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{s_1}\partial/\partial x^1 \\ \frac{1}{s_2}\partial/\partial x^2 \\ \frac{1}{s_3}\partial/\partial x^3 \end{pmatrix}$$

$\longrightarrow$ Modified Maxwell equation (frequency domain):

$$\begin{aligned} \tilde{\nabla} \cdot \varepsilon\vec{E} &= 0 \\ \tilde{\nabla} \cdot \mu\vec{H} &= 0 \\ \tilde{\nabla} \times \vec{E} &= -i\omega\mu\vec{H} \\ \tilde{\nabla} \times \vec{H} &= i\omega\varepsilon\vec{E} \end{aligned}$$

We have seen that this complex coordinate stretching

- offers a PML with great accuracy.
- can be easily used for 1, 2 or 3 dimensions.
- involves a modification of Maxwell's equations!
  $\Rightarrow$ can't be implemented easily in existing FEM code.

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Solution: use material constants ε and μ to provide the needed additional degrees of freedom.

$\Rightarrow$ Maxwellian PML

Maxwellian PML

Maxwell's equations in time harmonic form:

$$
\begin{aligned}
\nabla \cdot \bar{\varepsilon}\vec{E} &= 0 \\
\nabla \cdot \bar{\mu}\vec{H} &= 0 \\
\nabla \times \vec{E} &= -i\omega\mu\vec{H} - \sigma_M \vec{H} \\
\nabla \times \vec{H} &= i\omega\varepsilon\vec{E} + \sigma_E \vec{E}
\end{aligned}
$$

with

σ_M : magnetic conductivity $\qquad$ σ_E : electric conductivity

$$
\begin{aligned}
\bar{\varepsilon} &= \varepsilon_0 \begin{pmatrix} \varepsilon_x + \frac{\sigma_E^x}{i\omega} & 0 & 0 \\ 0 & \varepsilon_y + \frac{\sigma_E^y}{i\omega} & 0 \\ 0 & 0 & \varepsilon_z + \frac{\sigma_E^z}{i\omega} \end{pmatrix} \\
\bar{\mu} &= \mu_0 \begin{pmatrix} \mu_x + \frac{\sigma_M^x}{i\omega} & 0 & 0 \\ 0 & \mu_y + \frac{\sigma_M^y}{i\omega} & 0 \\ 0 & 0 & \mu_z + \frac{\sigma_M^z}{i\omega} \end{pmatrix}
\end{aligned}
$$

Nessecary condition for a PML:

Impedance matching : $Z = \sqrt{\frac{\mu}{\varepsilon}} = Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$

$$\Rightarrow \frac{\bar{\mu}}{\mu_0} = \frac{\bar{\varepsilon}}{\varepsilon_0} = \Lambda = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

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$\Rightarrow$ Maxwell's equations reduce to

$$\begin{aligned} \nabla \cdot \Lambda \vec{E} &= 0 \\ \nabla \cdot \Lambda \vec{H} &= 0 \\ \nabla \times \vec{E} &= -i\omega\mu_0 \Lambda \vec{H} \\ \nabla \times \vec{H} &= i\omega\varepsilon_0 \Lambda \vec{E} \end{aligned}$$

This equations lead to plane waves

$$\begin{aligned}\vec{E}(\vec{r},t) &= \vec{\mathcal{E}} e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \\ \vec{H}(\vec{r},t) &= \vec{\mathcal{H}} e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\end{aligned}$$

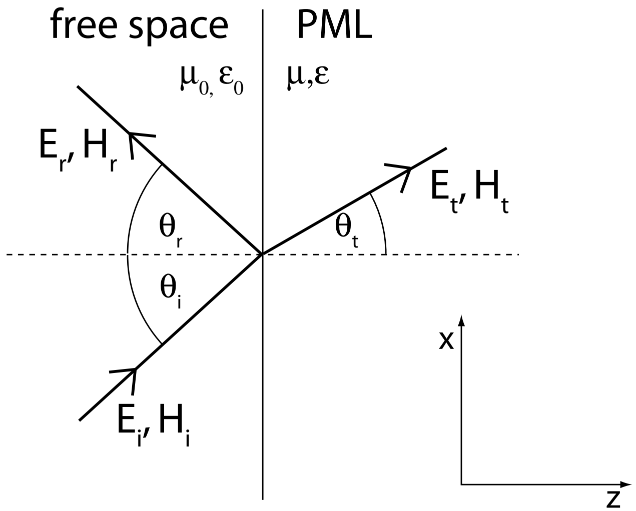
with the dispersion relation

$$\frac{k_x^2}{bc} + \frac{k_y^2}{ac} + \frac{k_z^2}{ab} = k_0^2 = \omega^2 \mu_0 \varepsilon_0,$$

which is the equation of an ellipsoid

$$\begin{aligned}k_x &= k_0 \sqrt{bc} \sin\theta \cos\phi \\ k_y &= k_0 \sqrt{ac} \sin\theta \sin\phi \\ k_z &= k_0 \sqrt{ab} \cos\theta\end{aligned}$$

Example



Dispersion relation:

$$
\begin{aligned}
k_x &= k_0\sqrt{bc}\sin\theta \\
k_y &= 0 \\
k_z &= k_0\sqrt{ab}\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\vec{E}_i, \vec{H}_i &\propto e^{-ik_0(\sin\theta_i x + \cos\theta_i z)} \\
\vec{E}_r, \vec{H}_r &\propto r \cdot e^{-ik_0(\sin\theta_r x + \cos\theta_r z)} \\
\vec{E}_t, \vec{H}_t &\propto t \cdot e^{-ik_0(\sqrt{bc}\sin\theta_t x + \sqrt{ab}\cos\theta_t z)}
\end{aligned}
$$

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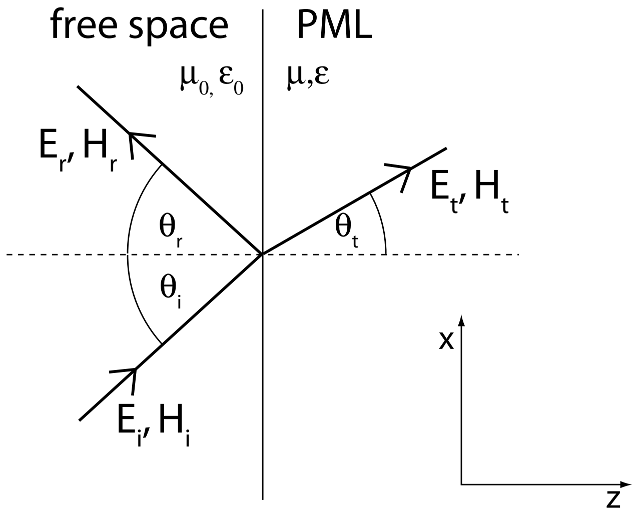

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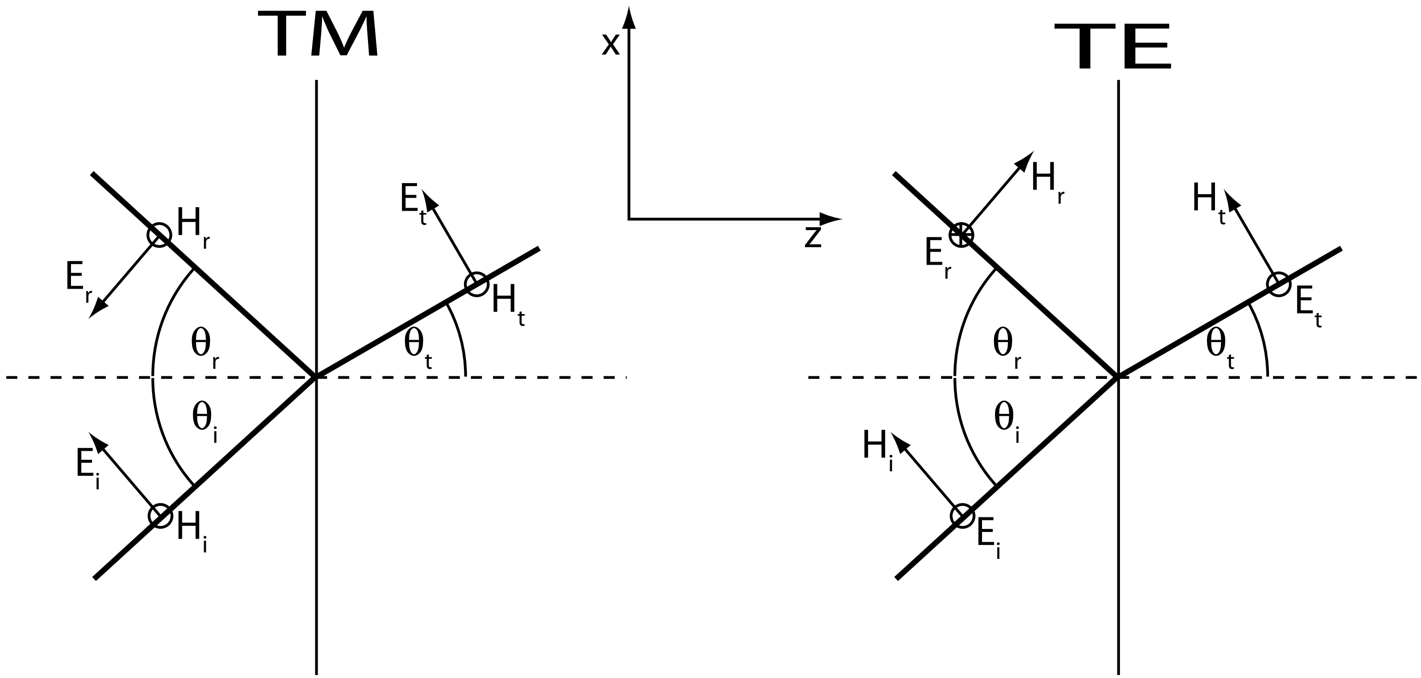
$$

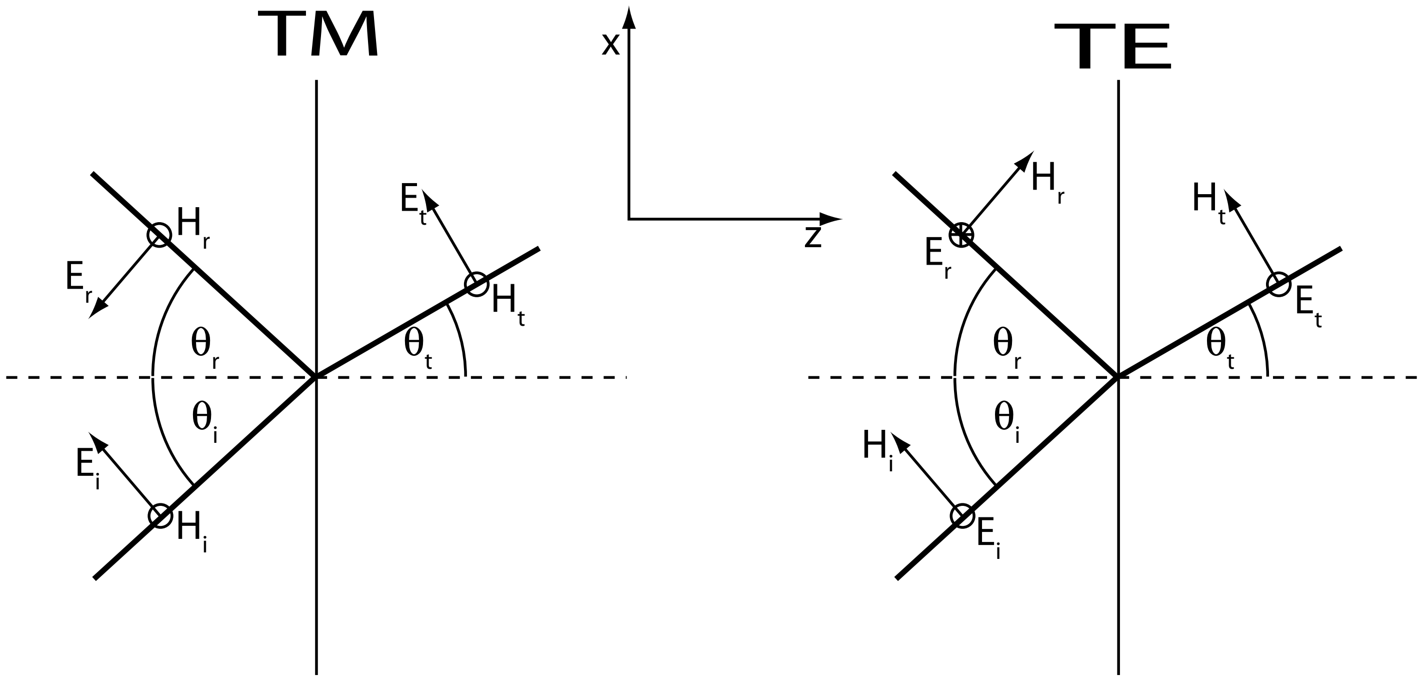
Continuity of the solutions on interface: $E_i + E_r = E_t$ and $H_i + H_r = H_t$
Phase matching yields a generalization of Snell's law

$$\sin\theta_i = \sin\theta_r = \sqrt{bc}\sin\theta_t$$

TM and TE waves



TM and TE waves



Reflection coefficients (using continuity of the solutions on the interface):

$$r^{TM} = \frac{\sqrt{\frac{b}{a}}\cos\theta_t - \cos\theta_i}{\cos\theta_i + \sqrt{\frac{b}{a}}\cos\theta_t} \qquad r^{TE} = \frac{\cos\theta_i - \sqrt{\frac{b}{a}}\cos\theta_t}{\cos\theta_i + \sqrt{\frac{b}{a}}\cos\theta_t}$$

Imposing

$$\sqrt{bc} = 1 \qquad \text{and} \qquad a = b$$

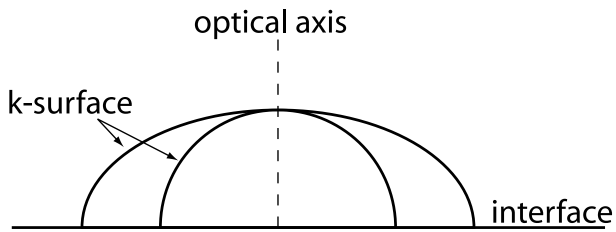
the interface will be perfectly reflectionless for any frequency, angle of incidence and polarization.
We now write $a = b = \frac{1}{c} = \alpha + i\beta$

$$\vec{E}_t(\vec{r}, t) = \vec{\mathcal{E}} e^{-k_0 \beta \cos\theta_t z} e^{-ik_0(\sin\theta_t x + \alpha\cos\theta_t z)} e^{i\omega t}$$

$$\Rightarrow \begin{array}{lll} \alpha & \leftrightarrow & \text{wave length in absorber} \\ \beta & \leftrightarrow & \text{rate of decay in absorber} \end{array}$$

penetration depth $\delta = \frac{1}{k_0 \beta \cos\theta_t}$

Physical Interpretation



- uniaxial crystal
- optical axis perpendicular to interface

- electric conductivity $\sigma_E = \omega\varepsilon_0 S$
- magnetic conductivity $\sigma_M = \omega\mu_0 S$

$$S = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\frac{\beta}{\alpha^2+\beta^2} \end{pmatrix}$$

z - component is negative $\rightarrow J_z = -\frac{\beta}{\alpha^2+\beta^2} E_z$

$\Rightarrow$ dependent sources in the material!

Summary Maxwellian PML

We now have found a PML formulation

- that uses an anisotropic material as an absorbing layer.
- that is similar but not equal to the techniques showed before.
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Problems and Questions remaining:

- Generalization to other geometries (e.g. cylindrical, spherical coordinates)?
- Link between the 2 PML formulations?
- Are there other PML formulations?

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Next Step:
Electromagnetics with differential forms

PML using differential forms

Non-Maxwellian PML formulation:

$$x^i \longmapsto \tilde{x}^i = \int_0^{x^i} s_i(\hat{x}^i) d\hat{x}^i$$

with

$$s_i(\hat{x}^i) = a_i(\hat{x}^i) + i\sigma_i(\hat{x}^i)/\omega$$

Re-Interpretation:

mapping on complex coordinates $\rightarrow$ change of metric

$$g_{ij} = \delta_{ij} \mapsto \tilde{g}_{ij} = g_{kl} \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^l}{\partial x^j} = \begin{pmatrix} (s_1)^2 & 0 & 0 \\ 0 & (s_2)^2 & 0 \\ 0 & 0 & (s_3)^2 \end{pmatrix}$$

General Case

- Consider the general orthogonal curvilinear case (u^1, u^2, u^3)
- g_{ij} is given in terms of the Lamé coefficients h_i: $g_{ij} = h_i^2(u^1, u^2, u^3) \cdot \delta_{ij}$
- Choose u^3 to be analytically continued: $u^3 \mapsto \tilde{u}^3 = \int_0^{u^3} s(\lambda) d\lambda$

$$\rightarrow \tilde{g}_{ij} = \begin{pmatrix} (\tilde{h}_1)^2 & 0 & 0 \\ 0 & (\tilde{h}_2)^2 & 0 \\ 0 & 0 & (\tilde{h}_3)^2 \end{pmatrix}$$

with $\tilde{h}_{1/2} = h_{1/2}(u^1, u^2, \tilde{u}^3)$ and $\tilde{h}_3 = s h_3(u^1, u^2, \tilde{u}^3)$.

Mapping forms to vectors

Given a metric $g_{ij} = (h_i)^2 \cdot \delta_{ij}$ there is a natural isomorphism mapping

- 1-forms to vectors

$$\Omega = \Omega_i du^i \qquad \overset{g_{ij}}{\longmapsto} \qquad \vec{\Omega} = \frac{\Omega_i}{h_i} \vec{u}^i$$

- 2-forms to axial vectors

$$\Phi = \Phi_i du^{[i+1]} \wedge du^{[i+2]} \qquad \overset{g_{ij}}{\longmapsto} \qquad \vec{\Phi} = \frac{\Phi_i}{h_{[i+1]} h_{[i+2]}} \vec{u}^i$$

with $\vec{u}^i$ the unit vector in u^i direction and $[i] \equiv i \mod 3$ for $i \neq 3$ and $[3] = 3$.

Maxwell's equations

Maxwell's Equations using differential forms (no sources!)

$$
\begin{aligned}
dE &= i\omega B \\
dH &= -i\omega D \\
dD &= 0 \\
dB &= 0
\end{aligned}
$$

- E,H : el., magn. field intensity 1-forms
- D,B : el., magn. flux density 2-forms
- d : exterior derivative, metric independent

$$
\text{d acts} \begin{cases} \text{on 1-forms } (\hat{=}\text{vectors}) & : \quad curl \\ \text{on 2-forms } (\hat{=}\text{axial vectors}) & : \quad div \end{cases}
$$

Constitutive Parameters

For differential forms, the constitutive parameters are given in terms of Hodge star operators:

$$
\begin{aligned}
D &= \star_e E \\
B &= \star_h H
\end{aligned}
$$

The Hodge Star operator

- establishes in the 3D case a natural isomorphism between the 1-forms E, H and the 2-forms D, B.
- depends on the metric
- for the euclidean metric is given though $\star dx = dydz$, $\star dy = dxdz$ and $\star dz = dxdy$.

Expressing the electric and magnetic 1-forms in terms of (u^1, u^2, u^3)

$$E = E_i h_i du^i \qquad\qquad H = H_i h_i du^i$$

The flux 2-forms become

$$D = \star_e(E_i h_i du^i) = \sum_j \varepsilon_{ij} E_j h_{[i+1]} h_{[i+2]} du^{[i+1]} \wedge du^{[i+2]}$$

$$B = \star_h(H_i h_i du^i) = \sum_j \mu_{ij} H_j h_{[i+1]} h_{[i+2]} du^{[i+1]} \wedge du^{[i+2]}$$

NB: the star operator depends on the metric!

Change of metric

Maxwell's equations under a change on the metric

$$\left.\begin{array}{rcl} d\tilde{E} & = & i\omega\tilde{B} \\ d\tilde{H} & = & -i\omega\tilde{D} \\ d\tilde{D} & = & 0 \\ d\tilde{B} & = & 0 \end{array}\right\}\text{same as before.}$$

$$\left.\begin{array}{rcl} \tilde{D} & = & \tilde{\star}_e\tilde{E} \\ \tilde{B} & = & \tilde{\star}_h\tilde{H} \end{array}\right\}\text{modified operators } \tilde{\star}_{e/h} \text{ defined by new metric.}$$

- The PML in the diff. forms language is unique and unifies the various PML formulations.
- The different formulations can be derived by a simple choice on how to map the forms to vector quantities.

The Maxwellian PML Formulation

Map from forms to corresponding dual vector quantities governed by original metric tensor (g_{ij}):

$$\tilde{E} = \tilde{E}_i \tilde{h}_i du^i \stackrel{(g_{ij})}{\longrightarrow} \vec{E}^m = E_i^m \vec{u}^i = \frac{\tilde{h}_i}{h_i} \tilde{E}_i \vec{u}^i$$

$$\tilde{D} \stackrel{(g_{ij})}{\longrightarrow} \vec{D}^m = D_i^m \vec{u}^i = \sum_j \frac{\tilde{h}_k \tilde{h}_l}{h_k h_l} \varepsilon_{ij} \tilde{E}_j \vec{u}^i$$

Modified constitutive tensors are given through

$$\vec{D}^m = \varepsilon_{PML} \cdot \vec{E}^m$$

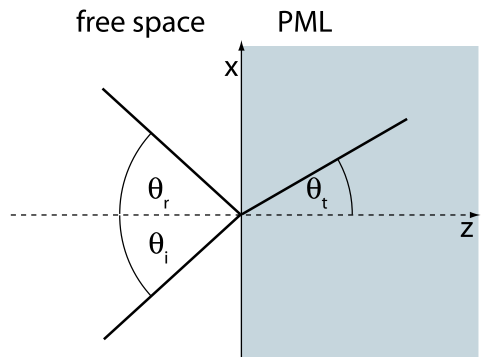
with

$$(\varepsilon_{PML})_{ij} = \frac{\tilde{h}_{[i+1]} \tilde{h}_{[i+2]}}{h_{[i+1]} h_{[i+2]}} \varepsilon_{ij} \frac{h_j}{\tilde{h}_j}$$

Example

free space:

$$h_1 = 1$$
$$h_2 = 1$$
$$h_3 = 1$$



Inside PML:

$$\tilde{h}_1 = 1$$
$$\tilde{h}_2 = 1$$
$$\tilde{h}_3 = s(z)$$

$$z \mapsto \tilde{z} = \int_0^z s(\lambda) d\lambda$$

$$\varepsilon_{ij} = \varepsilon_0 \cdot \delta_{ij} \Rightarrow (\varepsilon_{PML})_{ii} = \frac{\tilde{h}_{[i+1]}\tilde{h}_{[i+2]}}{h_{[i+1]}h_{[i+2]}} \varepsilon_0 \frac{h_i}{\tilde{h}_i}$$

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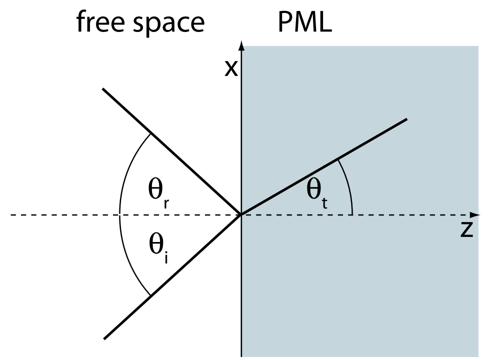

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$$\Rightarrow \varepsilon_{PML} = \varepsilon_0 \begin{pmatrix} s(z) & 0 & 0 \\ 0 & s(z) & 0 \\ 0 & 0 & \frac{1}{s(z)} \end{pmatrix}$$

In accordance with the result derived before!

Non-Maxwellian PML Formulation

Map from forms to corresponding dual vector quantities governed by modified, complex metric tensor $(\tilde{g}_{ij})$:

$$\tilde{E} = \tilde{E}_i \tilde{h}_i du^i \overset{(\tilde{g}_{ij})}{\longrightarrow} \vec{E}^c = E_i^c \vec{u}^i = \tilde{E}_i \vec{u}^i$$

$$\tilde{D} \overset{(\tilde{g}_{ij})}{\longrightarrow} \vec{D}^c = D_i^c \vec{u}^i = \sum_j \varepsilon_{ij} \tilde{E}_j \vec{u}^i$$

In contrary to the Maxwellian formulation, we obtain that

- the constitutive relations stay the same: $\vec{D}^c = \varepsilon \vec{E}^c$ and $\vec{B}^c = \mu \vec{H}^c$
- Maxwell's equations are modified to add additional degrees of freedom. In the Cartesian case, we obtain

$$\begin{aligned} \tilde{\nabla} \cdot \varepsilon \vec{E} &= 0 \qquad & \tilde{\nabla} \times \vec{E} &= -i\omega\mu\vec{H} \\ \tilde{\nabla} \cdot \mu \vec{H} &= 0 \qquad & \tilde{\nabla} \times \vec{H} &= i\omega\varepsilon\vec{E} \end{aligned}$$

New Classes of PML

Other choices of metrics $(\hat{g}_{ij})$ are also possible: e.g. hybridizations:

- $(\hat{g}_{ij}) = \alpha(g_{ij}) + \beta(\tilde{g}_{ij})$
- $(\hat{g}_{ij}) = \sum_{k=1}^{3} (g_{ik})^{\alpha} (\tilde{g}_{kj})^{\beta}$

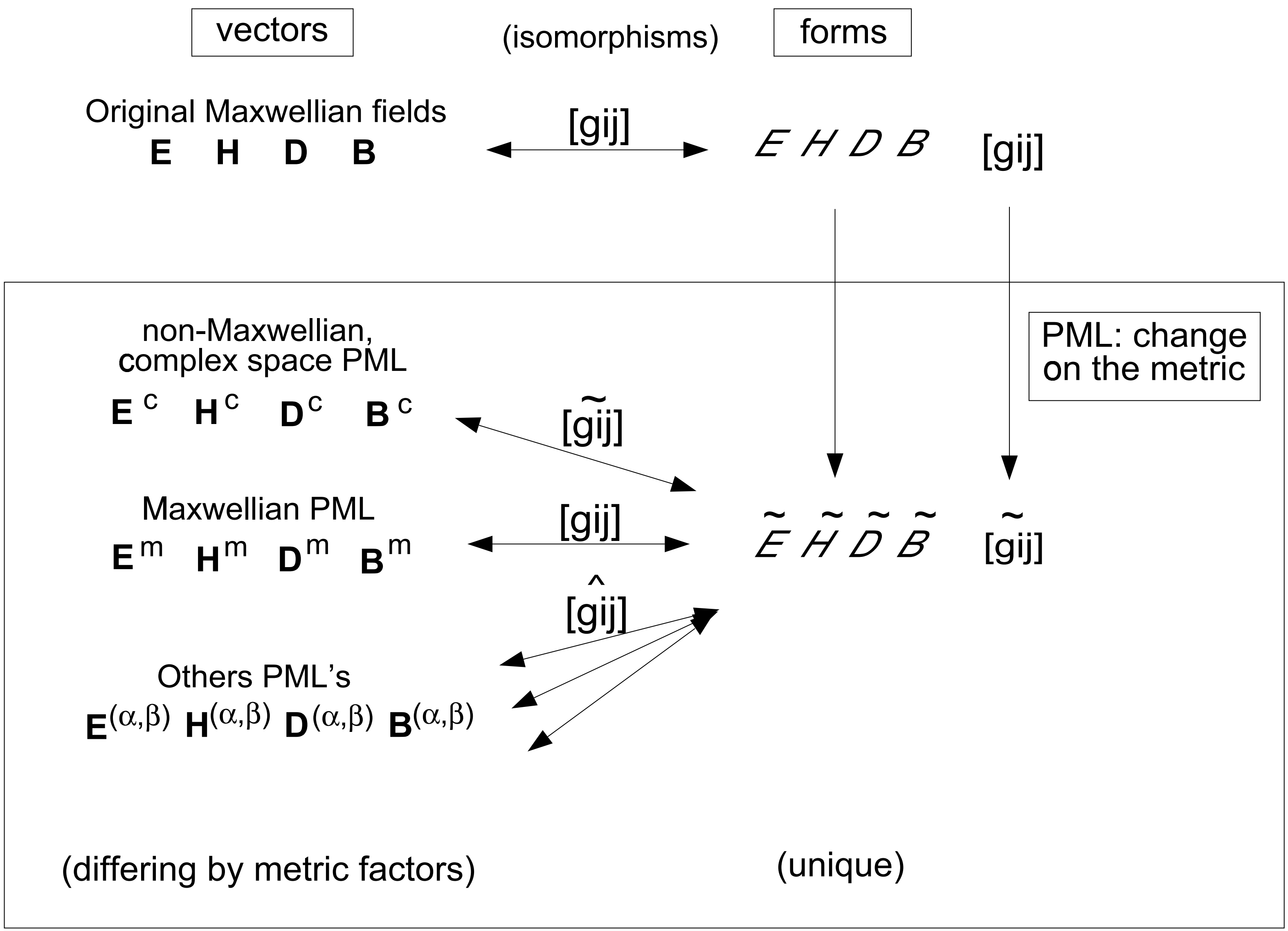
The second choice leads to

$$\begin{aligned} \vec{E}^{(\alpha,\beta)} &= E_i^{(\alpha,\beta)} \vec{u}^i = \frac{\tilde{h}_i^{1-\beta}}{h_i^{\alpha}} \tilde{E}_i \vec{u}^i \\ \vec{D}^{(\alpha,\beta)} &= D_i^{(\alpha,\beta)} \vec{u}^i = \sum_j \frac{\tilde{h}_{[i+1]}^{1-\beta} \tilde{h}_{[i+2]}^{1-\beta}}{h_{[i+1]}^{\alpha} h_{[i+2]}^{\alpha}} \varepsilon_{ij} \tilde{E}_j \vec{u}^i \end{aligned}$$

and a permittivity

$$\varepsilon_{ij}^{(\alpha,\beta)} = \frac{\tilde{h}_{[i+1]}^{1-\beta} \tilde{h}_{[i+2]}^{1-\beta}}{h_{[i+1]}^{\alpha} h_{[i+2]}^{\alpha}} \varepsilon_{ij} \frac{h_j^{\alpha}}{\tilde{h}_j^{1-\beta}}$$

Summary


vectors
(isomorphisms)
forms
Original Maxwellian fields
E H D B
[gij]
E H D B
[gij]
PML: change on the metric
non-Maxwellian, complex space PML
E c H c D c B c
[gij]
Maxwellian PML
E m H m D m B m
[gij]
E H D B
[gij]
[gij]
Others PML's
E(α,β) H(α,β) D(α,β) B(α,β)
(differing by metric factors)
(unique)

Conclusion

1

- Change of variables $\longrightarrow$ change of constitutive parameters
- No change on Maxwell's equations!
- tedious calculation

2

Differential forms provide

- a method independent of the field equations.
- an elegant way to generalize a PML for different geometries.
- a unique formulation for a PML.
  (different formulations correspond to different mappings)