An adaptive PML technique for time-harmonic scattering problems

Following a paper by Zhiming Chen and Xuezhe Liu

Manuel Largo

Overview

- Introduction, Hankel functions

Overview

- Introduction, Hankel functions
- PML formulation

Overview

- Introduction, Hankel functions
- PML formulation
- Finite Elements and the Main Theorem

Overview

- Introduction, Hankel functions
- PML formulation
- Finite Elements and the Main Theorem
- Implementation and Examples

First Part

INTRODUCTION AND HANKEL FUNCTIONS

Task

We want to show how we can adapt finite element mesh size.

Task

We want to show how we can adapt finite element mesh size.

To do so, we need an a posteriori error estimate to control the error we make when discretizing space.

Task

We want to show how we can adapt finite element mesh size.

To do so, we need an a posteriori error estimate to control the error we make when discretizing space.

We extend the idea of using a posteriori error estimates to determine the PML parameters and propose an adaptive PML technique for solving the Helmholtz-type scattering problem.

Task

We want to show how we can adapt finite element mesh size.

To do so, we need an a posteriori error estimate to control the error we make when discretizing space.

We extend the idea of using a posteriori error estimates to determine the PML parameters and propose an adaptive PML technique for solving the Helmholtz-type scattering problem.

We will first introduce and prove some error estimates, later construct an algorithm to adapt mesh size with a posteriori error control.

Scattering problem

So, lets derive a PML technique for solving Helmholtz-type scattering problems with perfectly conducting boundary.

Scattering problem

So, lets derive a PML technique for solving Helmholtz-type scattering problems with perfectly conducting boundary.

Let $D \in \mathbb{R}^2$ denote the bounded domain (scatterer) with boundary Γ_D, $g \in H^{-1/2}(\Gamma_D)$ determined by the incoming wave, $\mathbf{n}$ the unit outer normal to Γ_D.

Scattering problem

So, lets derive a PML technique for solving Helmholtz-type scattering problems with perfectly conducting boundary.

Let $D \in \mathbb{R}^2$ denote the bounded domain (scatterer) with boundary Γ_D, $g \in H^{-1/2}(\Gamma_D)$ determined by the incoming wave, $\mathbf{n}$ the unit outer normal to Γ_D.

Helmholtz-type scattering problem (constant k):

$$\begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D} \\ \frac{\partial u}{\partial \mathbf{n}} &= -g \quad \text{on } \Gamma_D \\ \sqrt{r}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) &\to 0 \ \text{ as } r = |x| \to \infty \end{aligned}$$

Hankel functions

First, consider the Bessel equation for functions of order ν:

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) y = 0, \qquad \nu \in \mathbb{C}.$$

Hankel functions

First, consider the Bessel equation for functions of order ν:

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) y = 0, \qquad \nu \in \mathbb{C}.$$

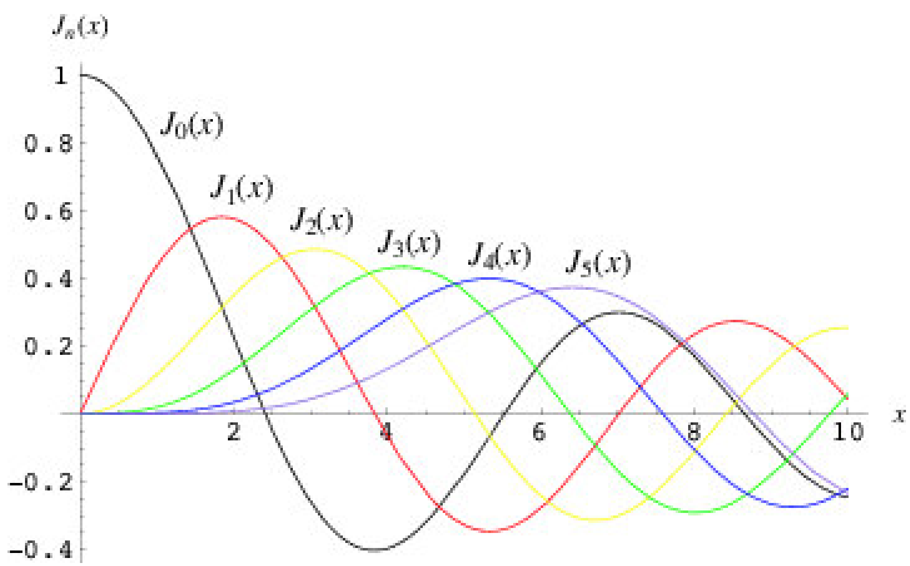
The so called Bessel function of the first kind $J_\nu(z)$ is defined as the solution to the Bessel differential equation with non singular values at the origin.

Hankel functions

First, consider the Bessel equation for functions of order ν:

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = 0, \qquad \nu \in \mathbb{C}.$$

The so called Bessel function of the first kind $J_\nu(z)$ is defined as the solution to the Bessel differential equation with non singular values at the origin.

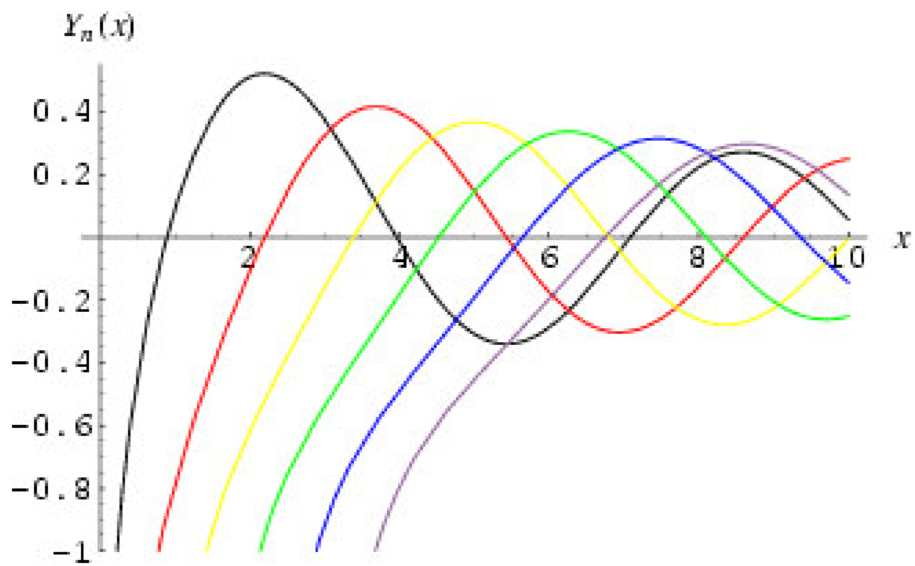


Hankel functions (cont)

The so called Bessel function of the second kind $Y_\nu(z)$ is defined as the solution to the Bessel differential equation with singular values at the origin.

Hankel functions (cont)

The so called Bessel function of the second kind $Y_\nu(z)$ is defined as the solution to the Bessel differential equation with singular values at the origin.



Hankel functions (cont)

We introduce now the Hankel function of the first kind and order ν $H_\nu^{(1)}(z), z \in \mathbb{C}$, and the Hankel function of the second kind and order ν $H_\nu^{(2)}(z)$, $z \in \mathbb{C}$, are defined by

$$
\begin{aligned}
H_\nu^{(1)}(z) &\equiv J_\nu(z) + \mathrm{i}Y_\nu(z), \\
H_\nu^{(2)}(z) &\equiv J_\nu(z) - \mathrm{i}Y_\nu(z).
\end{aligned}
$$

Hankel functions (cont)

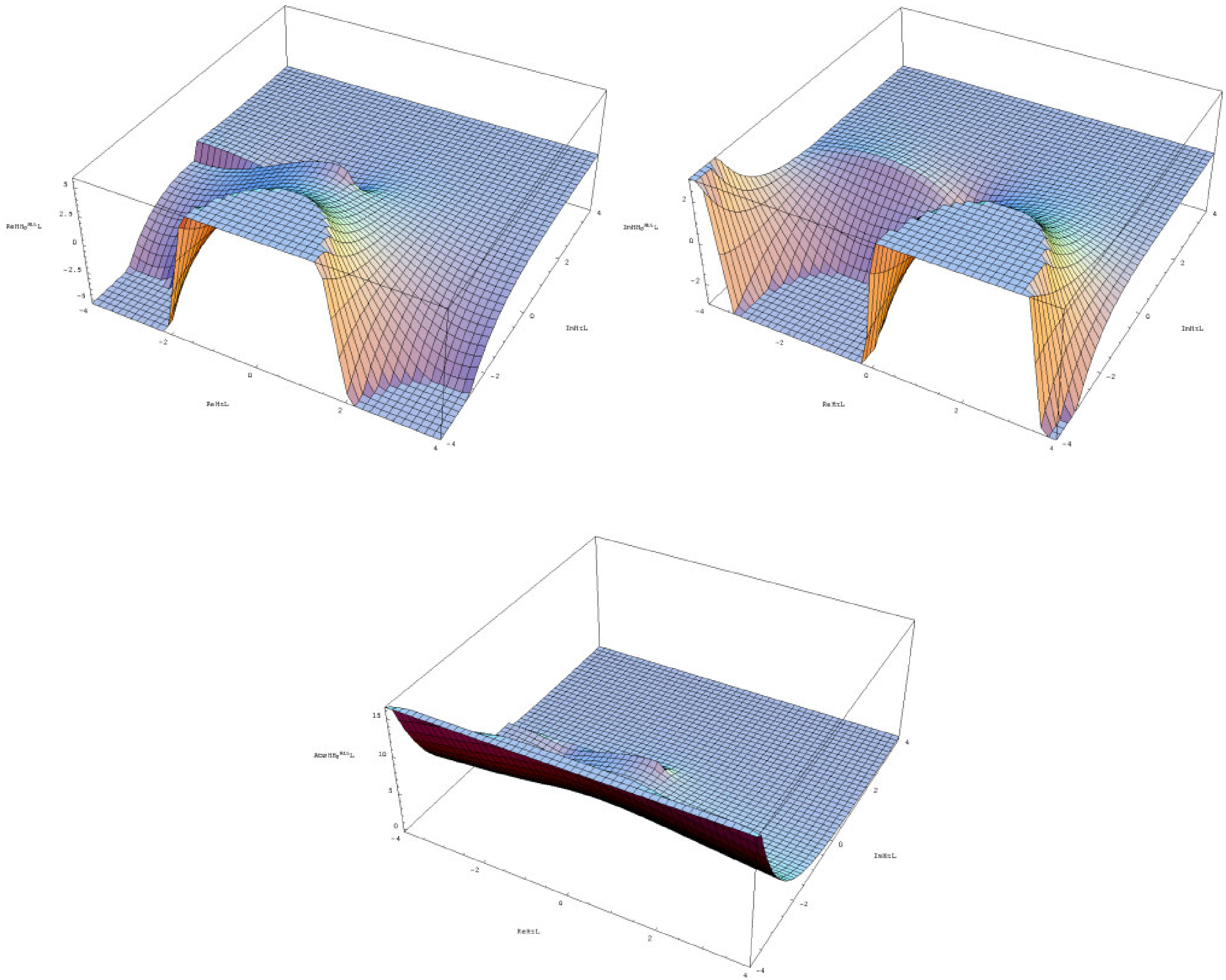
We introduce now the Hankel function of the first kind and order ν $H_\nu^{(1)}(z), z \in \mathbb{C}$, and the Hankel function of the second kind and order ν $H_\nu^{(2)}(z)$, $z \in \mathbb{C}$, are defined by

$$\begin{aligned} H_\nu^{(1)}(z) &\equiv J_\nu(z) + \mathrm{i}Y_\nu(z), \\ H_\nu^{(2)}(z) &\equiv J_\nu(z) - \mathrm{i}Y_\nu(z). \end{aligned}$$

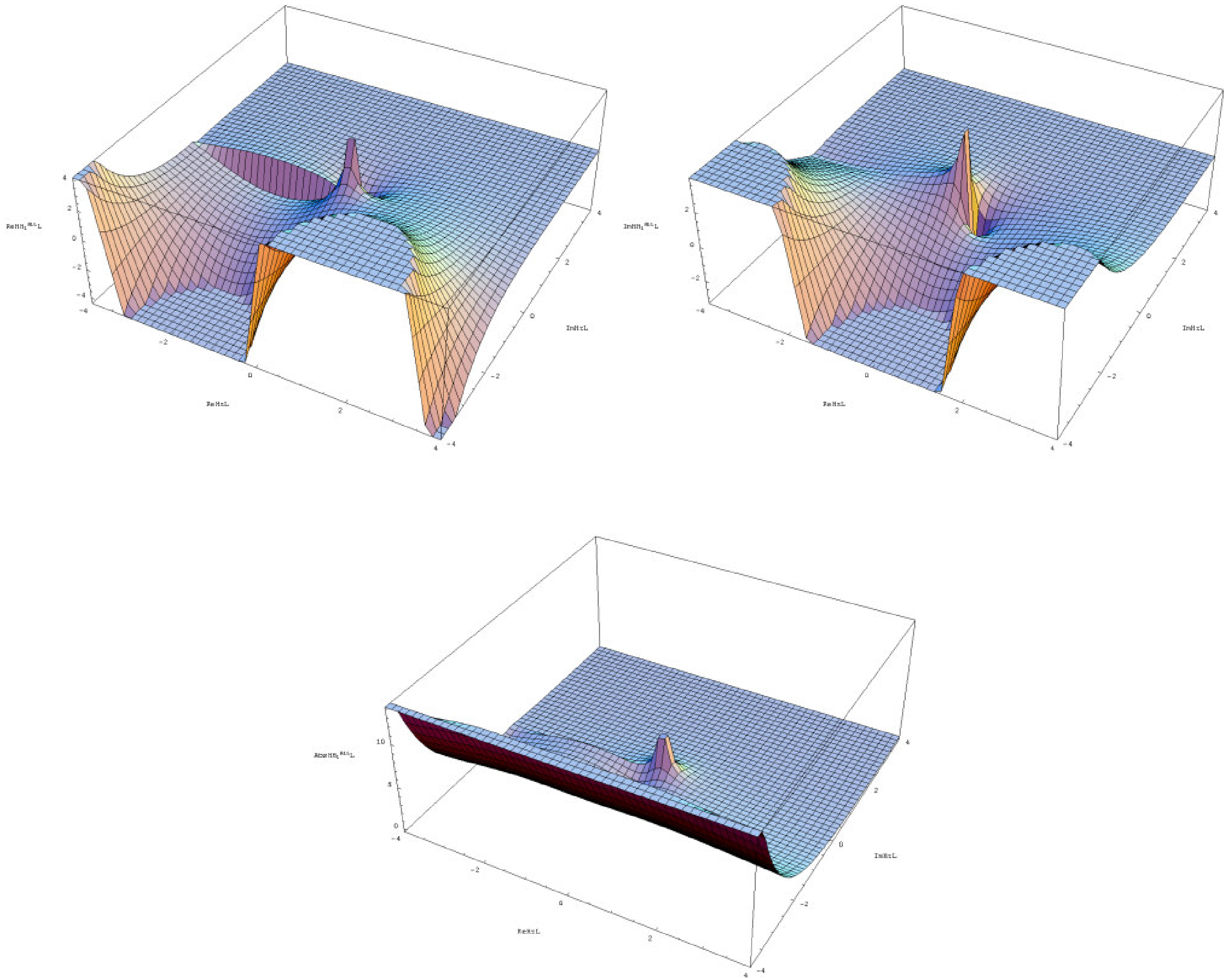
Asymptotic behaviour:

$$\begin{aligned} H_\nu^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \\ H_\nu^{(2)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}. \end{aligned}$$

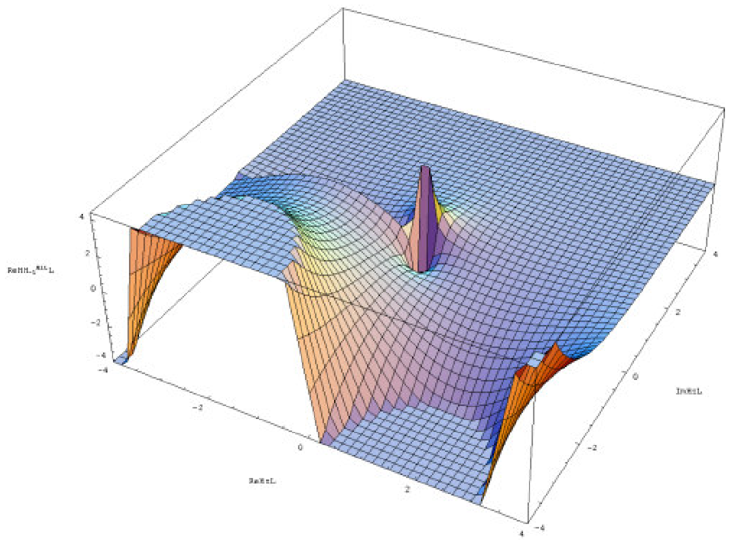
Hankel functions, $H_0^{(1)}$

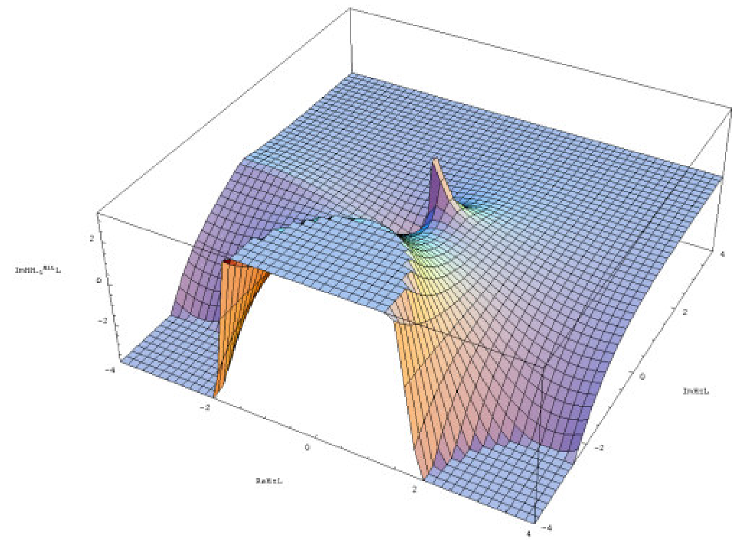


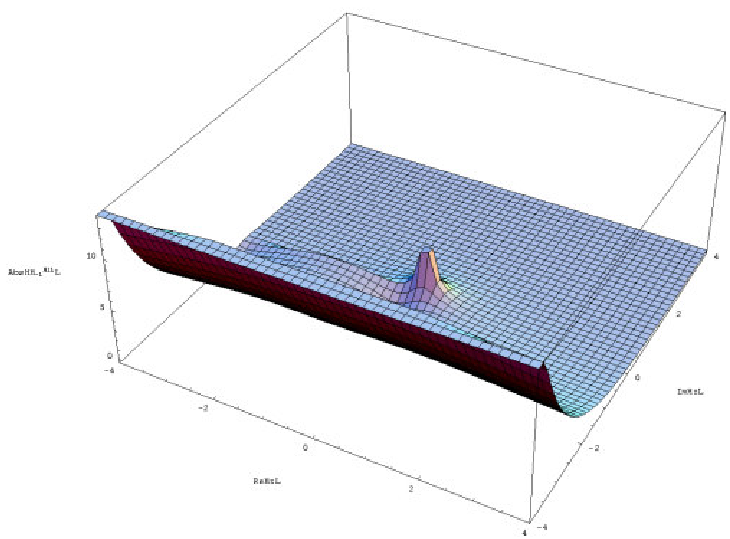
Hankel functions, $H_1^{(1)}$



Hankel functions, $H_{-1}^{(1)}$







Lemma 1

Lemma 1:

For any $\nu \in \mathbb{R}, z \in \mathbb{C}_{++} = \{z \in \mathbb{C} : \Im(z) \geq 0, \Re(z) \geq 0\}$, and $\Theta \in \mathbb{R}$ such that $0 < \Theta \leq |z|$, we have

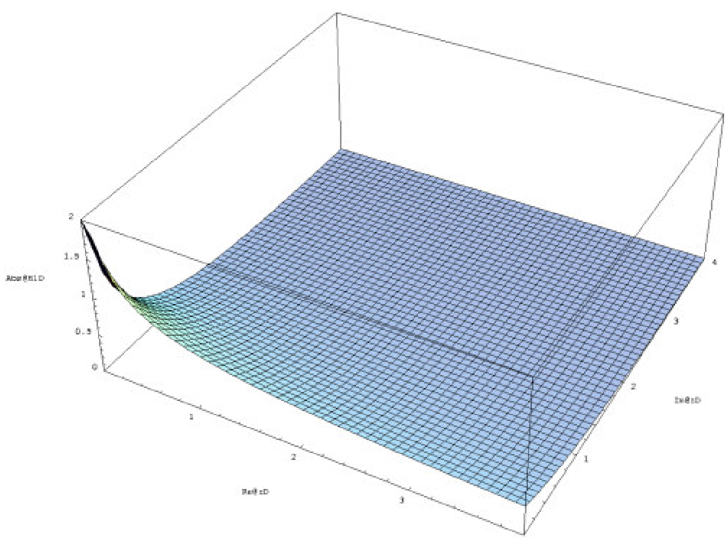
$$|H_\nu^{(1)}(z)| \leq e^{-\Im(z)\sqrt{1-\frac{\Theta^2}{|z|^2}}} |H_\nu^{(1)}(\Theta)|$$

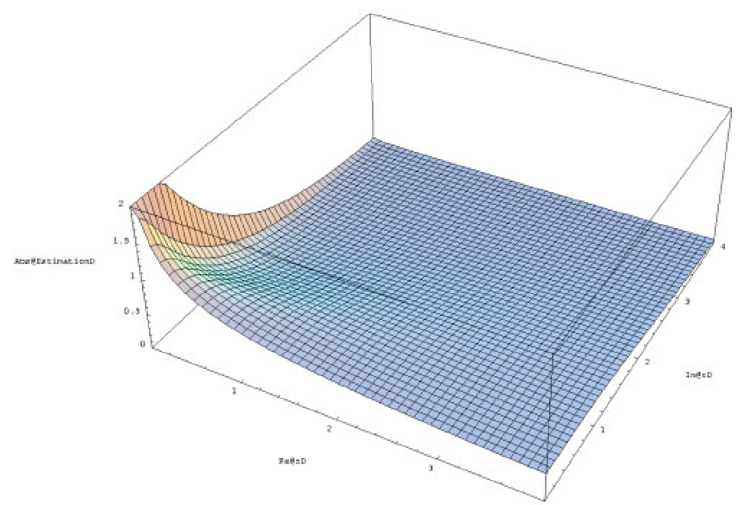
Lemma 1

Lemma 1:

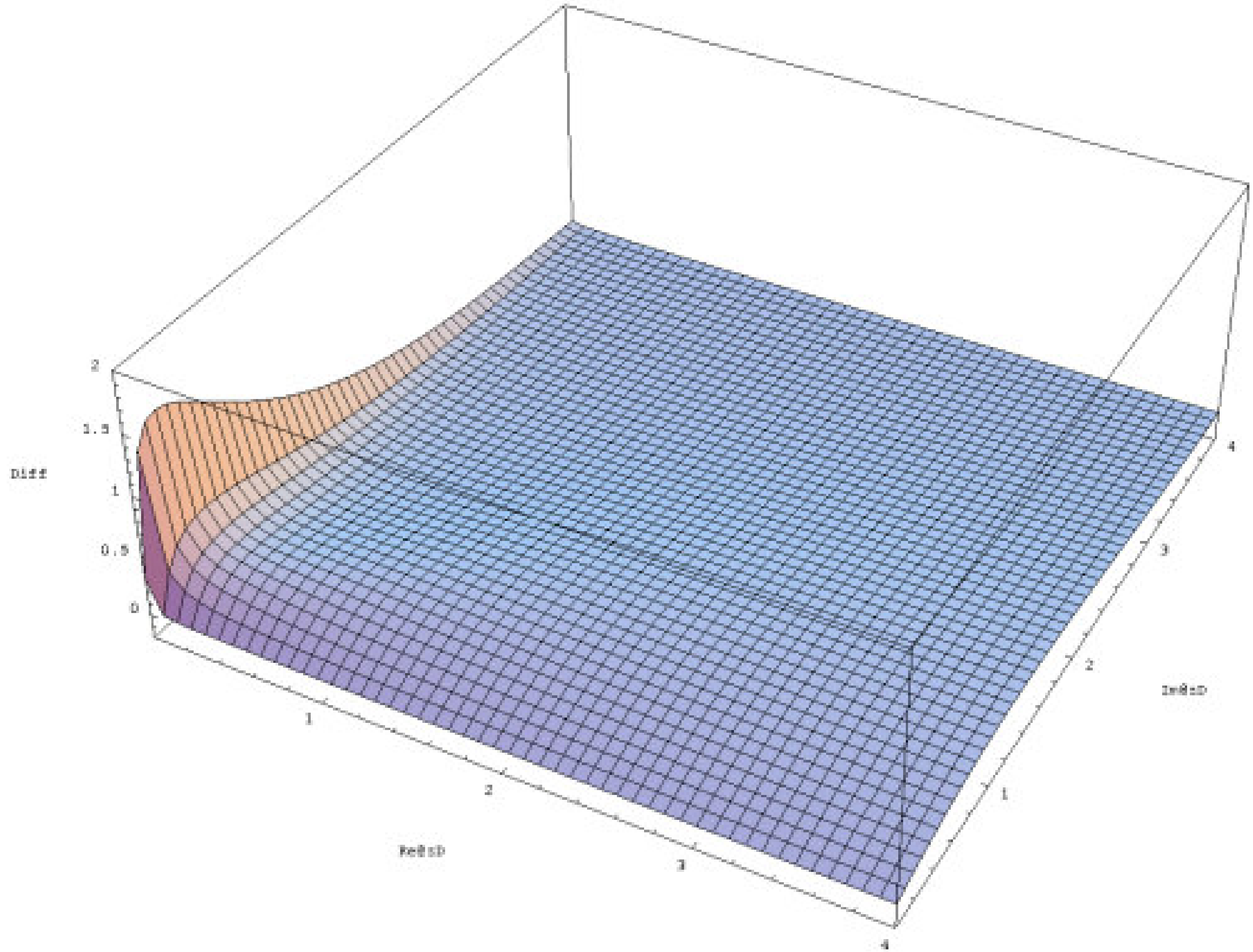
For any $\nu \in \mathbb{R}, z \in \mathbb{C}_{++} = \{z \in \mathbb{C} : \Im(z) \geq 0, \Re(z) \geq 0\}$, and $\Theta \in \mathbb{R}$ such that $0 < \Theta \leq |z|$, we have

$$|H_\nu^{(1)}(z)| \leq e^{-\Im(z)\sqrt{1-\frac{\Theta^2}{|z|^2}}} |H_\nu^{(1)}(\Theta)|$$





Lemma 1 (cont.)



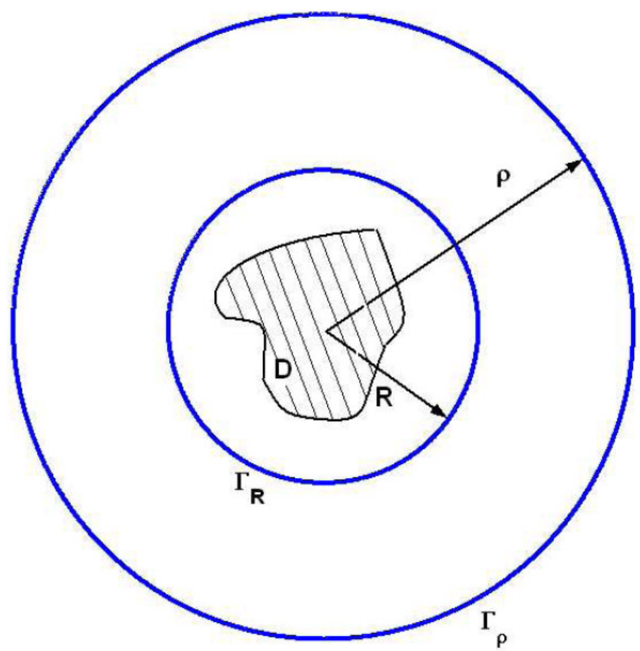
Second Part

PML FORMULATION

Setup

Let the scatterer D be contained in the interior of the circle $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$, and $\Omega_R = B_R \backslash \bar{D}$.

We now surround the domain Ω_R with a PML layer $\Omega^{\mathrm{PML}} = \{x \in \mathbb{R}^2 : R < |x| < \rho\}$.



The PML formulation

Look at the domain $\mathbb{R}^2\backslash\bar{B}_R$. The solution u of the scattering problem can be written under the polar coordinates as follows:

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)}\hat{u}_n e^{\mathrm{i}n\theta}, \qquad \hat{u}_n = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)e^{-\mathrm{i}n\theta}d\theta.$$

$H_n^{(1)}$ denotes the just discussed Hankel function of the first kind and order n. It can be shown that this series converges uniformly for $r > R$.

Dirichlet-to-Neumann operator

We now introduce the so called Dirichlet-to-Neumann operator $T : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$, where $\Gamma_R = \partial B_R$. It is definied as follows: for any $f \in H^{1/2}(\Gamma_R)$,

$$Tf = \sum_{n\in\mathbb{Z}} k \frac{H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} \hat{f}_n e^{\mathrm{i}n\theta}, \qquad \hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f e^{-\mathrm{i}n\theta} d\theta.$$

Dirichlet-to-Neumann operator

We now introduce the so called Dirichlet-to-Neumann operator $T : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$, where $\Gamma_R = \partial B_R$. It is definied as follows: for any $f \in H^{1/2}(\Gamma_R)$,

$$Tf = \sum_{n\in\mathbb{Z}} k \frac{H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} \hat{f}_n e^{\mathrm{i}n\theta}, \qquad \hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f e^{-\mathrm{i}n\theta} d\theta.$$

Looking at the representation of the solution u in polar coordinates:

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)} \hat{u}_n e^{\mathrm{i}n\theta}, \qquad \hat{u}_n = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta) e^{-\mathrm{i}n\theta} d\theta,$$

it is obvious that it satisfies

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\Gamma_R} = Tu.$$

Reformulation

Let $a : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C}$ be the sesquilinear form

$$a(\varphi, \psi) = \int_{\Omega_R} \left(\nabla\varphi \cdot \nabla\bar{\psi} - k^2 \varphi\bar{\psi} \right) dx - \langle T\varphi, \psi \rangle_{\Gamma_R}.$$

Reformulation

Let $a : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C}$ be the sesquilinear form

$$a(\varphi, \psi) = \int_{\Omega_R} \left(\nabla\varphi \cdot \nabla\bar{\psi} - k^2\varphi\bar{\psi}\right) dx - \langle T\varphi, \psi\rangle_{\Gamma_R}.$$

Given $g \in H^{-1/2}(\Gamma_R)$, find $u \in H^1(\Gamma_R)$ such that

$$a(u, \psi) = \langle g, \psi\rangle_{\Gamma_D} \qquad \forall\psi \in H^1(\Omega_R), \mu > 0.$$

Reformulation

Let $a : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C}$ be the sesquilinear form

$$a(\varphi, \psi) = \int_{\Omega_R} \left(\nabla\varphi \cdot \nabla\bar{\psi} - k^2\varphi\bar{\psi}\right) dx - \langle T\varphi, \psi\rangle_{\Gamma_R}.$$

Given $g \in H^{-1/2}(\Gamma_R)$, find $u \in H^1(\Gamma_R)$ such that

$$a(u, \psi) = \langle g, \psi\rangle_{\Gamma_D} \qquad \forall\psi \in H^1(\Omega_R), \mu > 0.$$

$$\sup_{0 \neq \psi \in H^1(\Omega_R)} \frac{|a(\varphi, \psi)|}{\|\psi\|_{H^1(\Omega_R)}} \geq \mu \|\varphi\|_{H^1(\Omega_R)} \qquad \forall\varphi \in H^1(\Omega_R).$$

PML formulation

Let $\alpha(r) = 1 + \mathrm{i}\sigma(r)$ be the PML model medium property with

$$\sigma \in C(\mathbb{R}), \qquad \sigma \geq 0, \qquad \text{and } \sigma = 0 \text{ for } r \leq R.$$

We denote by $\tilde{r}$ the complex radius defined by

$$\tilde{r} = \tilde{r}(r) = \begin{cases} r & \text{if } r \leq R, \\ \int_0^r \alpha(t)dt = r\beta(r) & \text{if } r \geq R. \end{cases}$$

PML formulation

Let $\alpha(r) = 1 + \mathrm{i}\sigma(r)$ be the PML model medium property with

$$\sigma \in C(\mathbb{R}), \qquad \sigma \geq 0, \qquad \text{and } \sigma = 0 \text{ for } r \leq R.$$

We denote by $\tilde{r}$ the complex radius defined by

$$\tilde{r} = \tilde{r}(r) = \begin{cases} r & \text{if } r \leq R, \\ \int_0^r \alpha(t)dt = r\beta(r) & \text{if } r \geq R. \end{cases}$$

Lets introduce now the PML equation:

$$\nabla \cdot (A \nabla w) + \alpha\beta k^2 w = 0 \qquad \text{in } \Omega^{\mathrm{PML}},$$

where $A = A(x)$ is a matrix which satisfies, in polar coordinates,

$$\nabla \cdot (A\nabla) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\beta r}{\alpha}\frac{\partial}{\partial r}\right) + \frac{\alpha}{\beta r^2}\frac{\partial^2}{\partial \theta^2}.$$

PML formulation (cont)

Now, the PML solution $\hat{u}$ in $\Omega_\rho = B_\rho \backslash \bar{D}$ is defined as the solution of the system

$$\begin{aligned} \nabla \cdot (A \nabla \hat{u}) + \alpha\beta k^2 \hat{u} &= 0 \quad \text{in } \Omega_\rho, \\ \frac{\partial \hat{u}}{\partial \mathbf{n}} &= -g \quad \text{on } \Gamma_D, \\ \hat{u} &= 0 \quad \text{on } \Gamma_\rho. \end{aligned}$$

PML formulation (cont)

Now, the PML solution $\hat{u}$ in $\Omega_\rho = B_\rho \backslash \bar{D}$ is defined as the solution of the system

$$\begin{aligned} \nabla \cdot (A \nabla \hat{u}) + \alpha\beta k^2 \hat{u} &= 0 \quad \text{in } \Omega_\rho, \\ \frac{\partial \hat{u}}{\partial \mathbf{n}} &= -g \quad \text{on } \Gamma_D, \\ \hat{u} &= 0 \quad \text{on } \Gamma_\rho. \end{aligned}$$

Again, we introduce the sesquilinear form $\hat{a} : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C}$ by

$$\hat{a}(\varphi, \psi) = \int_{\Omega_R} (A \nabla \varphi \cdot \nabla \bar{\psi} - k^2 \alpha\beta \varphi \bar{\psi}) dx - \langle \hat{T} \varphi, \psi \rangle_{\Gamma_R},$$

and

$$\hat{a}(\hat{u}, \psi) = \langle g, \psi \rangle_{\Gamma_D} \qquad \forall \psi \in H^1(\Omega_R).$$

PML formulation (cont)

Similar to the previous problem, we can reformulate the problem in the bounded domain Ω_R by imposing the boundary condition

$$\left.\frac{\partial \hat{u}}{\partial \mathbf{n}}\right|_{\Gamma_R} = \hat{T}\hat{u},$$

where $\hat{T} : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ is defined as follows: given $f \in H^{1/2}(\Gamma_R)$,

$$\hat{T}f = \left.\frac{\partial \zeta}{\partial \mathbf{n}}\right|_{\Gamma_R},$$

where $\zeta \in H^1(\Omega^{\mathrm{PML}})$ satisfies

$$\begin{aligned} \nabla \cdot (A\nabla\zeta) + \alpha\beta k^2 \zeta &= 0 \quad \text{in } \Omega^{\mathrm{PML}}, \\ \zeta &= f \quad \text{on } \Gamma_R, \\ \zeta &= 0 \quad \text{on } \Gamma_\rho. \end{aligned}$$

The PML equation in the layer

Lets look now at the Dirichlet problem in the PML layer Ω^{PML} only: The solution w solves

$$\begin{aligned} \nabla \cdot (A \nabla w) + \alpha\beta k^2 w &= 0 \quad \text{in } \Omega^{\mathrm{PML}}, \\ w &= 0 \quad \text{on } \Gamma_R, \\ w &= q \quad \text{on } \Gamma_\rho. \end{aligned}$$

where $q \in H^{1/2}(\Gamma_\rho)$.

The PML equation in the layer

Lets look now at the Dirichlet problem in the PML layer Ω^{PML} only: The solution w solves

$$
\begin{aligned}
\nabla \cdot (A \nabla w) + \alpha\beta k^2 w &= 0 \quad \text{in } \Omega^{\mathrm{PML}}, \\
w &= 0 \quad \text{on } \Gamma_R, \\
w &= q \quad \text{on } \Gamma_\rho.
\end{aligned}
$$

where $q \in H^{1/2}(\Gamma_\rho)$. With $\hat{b} : H^1(\Omega^{\mathrm{PML}}) \times H^1(\Omega^{\mathrm{PML}}) \to \mathbb{C}$ defined to be

$$
\hat{b}(\varphi, \psi) = \int_R^\rho \int_0^{2\pi} \left(\frac{\beta r}{\alpha} \frac{\partial \varphi}{\partial r} \frac{\partial \bar{\psi}}{\partial r} + \frac{\alpha}{\beta r} \frac{\partial \varphi}{\partial \theta} \frac{\partial \bar{\psi}}{\partial \theta} - \alpha\beta k^2 r \varphi \bar{\psi} \right) dr d\theta,
$$

we can write down the weak formulation for this problem:
given $q \in H^{1/2}(\Gamma_\rho)$, find $w \in H^1(\Omega^{\mathrm{PML}})$ such that $w = 0$ on Γ_R, $w = q$ on Γ_ρ, and

$$
\hat{b}(w, \varphi) = 0 \qquad \forall \varphi \in H_0^1(\Omega^{\mathrm{PML}}).
$$

Medium property

We make the following assumption for the fictitious medium property σ:

(H1): $\sigma = \sigma_0 \left(\frac{r-R}{\rho-R}\right)^m$ for some $\sigma_0 > 0$ and $m \in \mathbb{N}$.

Medium property

We make the following assumption for the fictitious medium property σ:

(H1): $\sigma = \sigma_0 \left(\frac{r-R}{\rho-R}\right)^m$ for some $\sigma_0 > 0$ and $m \in \mathbb{N}$.

We know that $\beta(r) = r^{-1} \int_0^r \alpha(t)dt$, and therefore $\beta(r) = 1 + \mathrm{i}\hat{\sigma}(r)$, where

$$\hat{\sigma}(r) = \frac{1}{r}\int_R^r \sigma(t)dt = \frac{\sigma_0}{m+1}\frac{r-R}{r}\left(\frac{r-R}{\rho-R}\right)^m .$$

Therefore, $\hat{\sigma} \le \sigma \; \forall r \ge R$.

Medium property

We make the following assumption for the fictitious medium property σ:

(H1): $\sigma = \sigma_0 \left(\frac{r-R}{\rho-R}\right)^m$ for some $\sigma_0 > 0$ and $m \in \mathbb{N}$.

We know that $\beta(r) = r^{-1} \int_0^r \alpha(t)dt$, and therefore $\beta(r) = 1 + \mathrm{i}\hat{\sigma}(r)$, where

$$\hat{\sigma}(r) = \frac{1}{r} \int_R^r \sigma(t)dt = \frac{\sigma_0}{m+1} \frac{r-R}{r} \left(\frac{r-R}{\rho-R}\right)^m .$$

Therefore, $\hat{\sigma} \leq \sigma \; \forall r \geq R$.

(H2) There exists a unique solution to the Dirichlet PML problem in the PML layer Ω^{PML}.

Theorem 1

We give the following theorem (without proof) as the main objective of this subsection:

Theorem 1

Let (H1)-(H2) be satisfied. There exists a constant $C > 0$ independent of k, R, ρ, and σ_0 such that the following estimates hold:

$$\begin{aligned} \||\alpha|^{-1}\nabla w\|_{L^2(\Omega^{\mathrm{PML}})} &\leq C\hat{C}^{-1}(1+kR)|\alpha_0|\|q\|_{H^{1/2}(\Gamma_\rho)}, \\ \left\|\frac{\partial w}{\partial \mathbf{n}}\right\|_{H^{-1/2}(\Gamma_R)} &\leq C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2\|q\|_{H^{1/2}(\Gamma_\rho)}. \end{aligned}$$

where $\alpha_0 = 1 + \mathrm{i}\sigma_0$.

Theorem 1

We give the following theorem (without proof) as the main objective of this subsection:

Theorem 1

Let (H1)-(H2) be satisfied. There exists a constant $C > 0$ independent of k, R, ρ, and σ_0 such that the following estimates hold:

$$\begin{aligned} \| |\alpha|^{-1} \nabla w \|_{L^2(\Omega^{\mathrm{PML}})} &\leq C \hat{C}^{-1} (1 + kR) |\alpha_0| \|q\|_{H^{1/2}(\Gamma_\rho)}, \\ \left\| \frac{\partial w}{\partial \mathbf{n}} \right\|_{H^{-1/2}(\Gamma_R)} &\leq C \hat{C}^{-1} (1 + kR)^2 |\alpha_0|^2 \|q\|_{H^{1/2}(\Gamma_\rho)}. \end{aligned}$$

where $\alpha_0 = 1 + \mathrm{i}\sigma_0$.

We will need these estimates later to prove the main theorem of this talk . . .

Propagation operator

To prove the convergence of the just considered PML problem to the original scattering problem, we need to introduce the propagation operator $P : H^{1/2}(\Gamma_R) \to H^{1/2}(\Gamma_\rho)$ defined as (Lassas and Somersalo):

$$P(f) = \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)}(k\tilde{\rho})}{H_n^{(1)}(kR)} \hat{f}_n e^{\mathrm{i}n\theta}, \qquad \hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f e^{-\mathrm{i}n\theta} d\theta.$$

Propagation operator

To prove the convergence of the just considered PML problem to the original scattering problem, we need to introduce the propagation operator $P : H^{1/2}(\Gamma_R) \to H^{1/2}(\Gamma_\rho)$ defined as (Lassas and Somersalo):

$$P(f) = \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)}(k\tilde{\rho})}{H_n^{(1)}(kR)} \hat{f}_n e^{\mathrm{i}n\theta}, \qquad \hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f e^{-\mathrm{i}n\theta} d\theta.$$

One can also show that

$$\|P(f)\|_{H^{1/2}(\Gamma_\rho)} \leq e^{-k\Im(\tilde{\rho})\sqrt{1-\frac{R^2}{|\tilde{\rho}|^2}}} \|f\|_{H^{1/2}(\Gamma_R)} \qquad \forall r \geq R.$$

D2N mapping

Lemma 2:

Let (H1)-(H2) be satisfied. Then, we have

$$\|Tf - \hat{T}f\|_{H^{-1/2}(\Gamma_R)} \leq C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 e^{-k\Im(\tilde{\rho})\sqrt{1-\frac{R^2}{|\tilde{\rho}|^2}}}\|f\|_{H^{1/2}(\Gamma_R)}.$$

D2N mapping

Lemma 2:

Let (H1)-(H2) be satisfied. Then, we have

$$\|Tf - \hat{T}f\|_{H^{-1/2}(\Gamma_R)} \leq C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 e^{-k\Im(\tilde{\rho})\sqrt{1-\frac{R^2}{|\tilde{\rho}|^2}}}\|f\|_{H^{1/2}(\Gamma_R)}.$$

Theorem 2:

Let again (H1)-(H2) be satisfied. Then, for sufficiently large $\sigma_0 > 0$, the PML problem has a unique solution $\hat{u} \in H^1(\Omega_\rho)$. Moreover, we have the following estimate:

$$\|u - \hat{u}\|_{H^1(\Omega_R)} \leq C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 e^{-k\Im(\tilde{\rho})\sqrt{1-\frac{R^2}{|\tilde{\rho}|^2}}}\|\hat{u}\|_{H^{1/2}(\Gamma_R)}.$$

Third Part

FINITE ELEMENTS AND THE MAIN THEOREM

The Finite Element Method (FEM)

Task: By discretization, transform a variational boundary value problem to a system of finite number of equations for real unknowns. I.e. transform the linear variational problem

$$u \in V : a(u, v) = f(v) \qquad \forall v \in V$$

to

$$u_N \in V_h : a(u_N, v_N) = f(v_N) \qquad \forall v_N \in V_h.$$

The Finite Element Method (FEM)

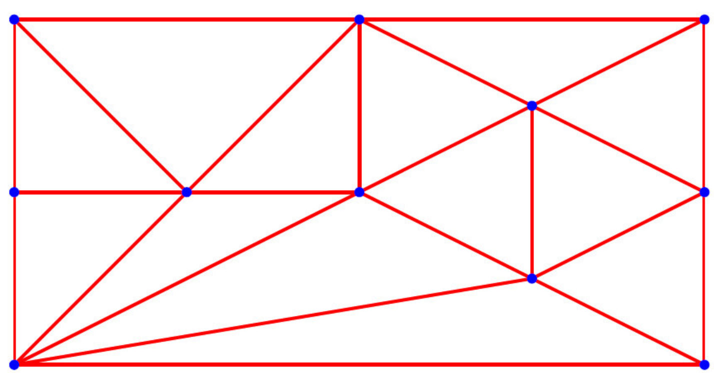
Task: By discretization, transform a variational boundary value problem to a system of finite number of equations for real unknowns. I.e. transform the linear variational problem

$$u \in V : a(u, v) = f(v) \quad \forall v \in V$$

to

$$u_N \in V_h : a(u_N, v_N) = f(v_N) \quad \forall v_N \in V_h.$$

Do it by triangulation of space Ω:



FEM basis functions

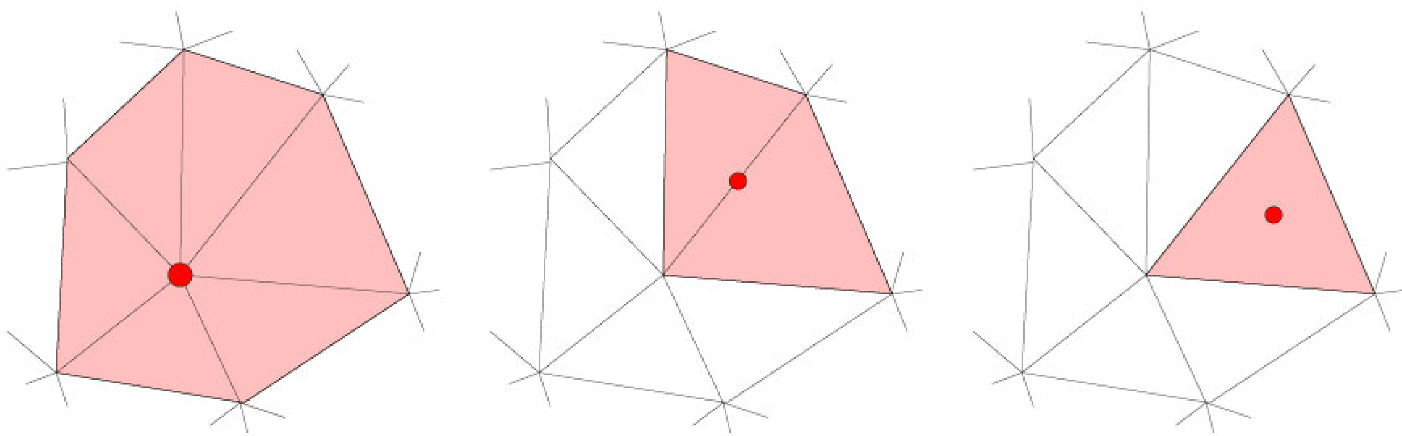
Basis functions $\phi_1, \dots, \phi_N$ for a finite element space V_h built on a mesh $\mathcal{M}_h$ satisfy:

- each ϕ_i associated with a single cell/edge/face/vertex of $\mathcal{M}_h$,
- $\mathrm{supp}(\phi_i) = \bigcup\{\bar{K} : K \in \mathcal{M}_h, p \subset \bar{K}\}$, if ϕ_i associated with cell/edge/face/vertex p.

FEM basis functions

Basis functions $\phi_1, \ldots, \phi_N$ for a finite element space V_h built on a mesh $\mathcal{M}_h$ satisfy:

- each ϕ_i associated with a single cell/edge/face/vertex of $\mathcal{M}_h$,
- $\mathrm{supp}(\phi_i) = \bigcup\{\bar{K} : K \in \mathcal{M}_h, p \subset \bar{K}\}$, if ϕ_i associated with cell/edge/face/vertex p.

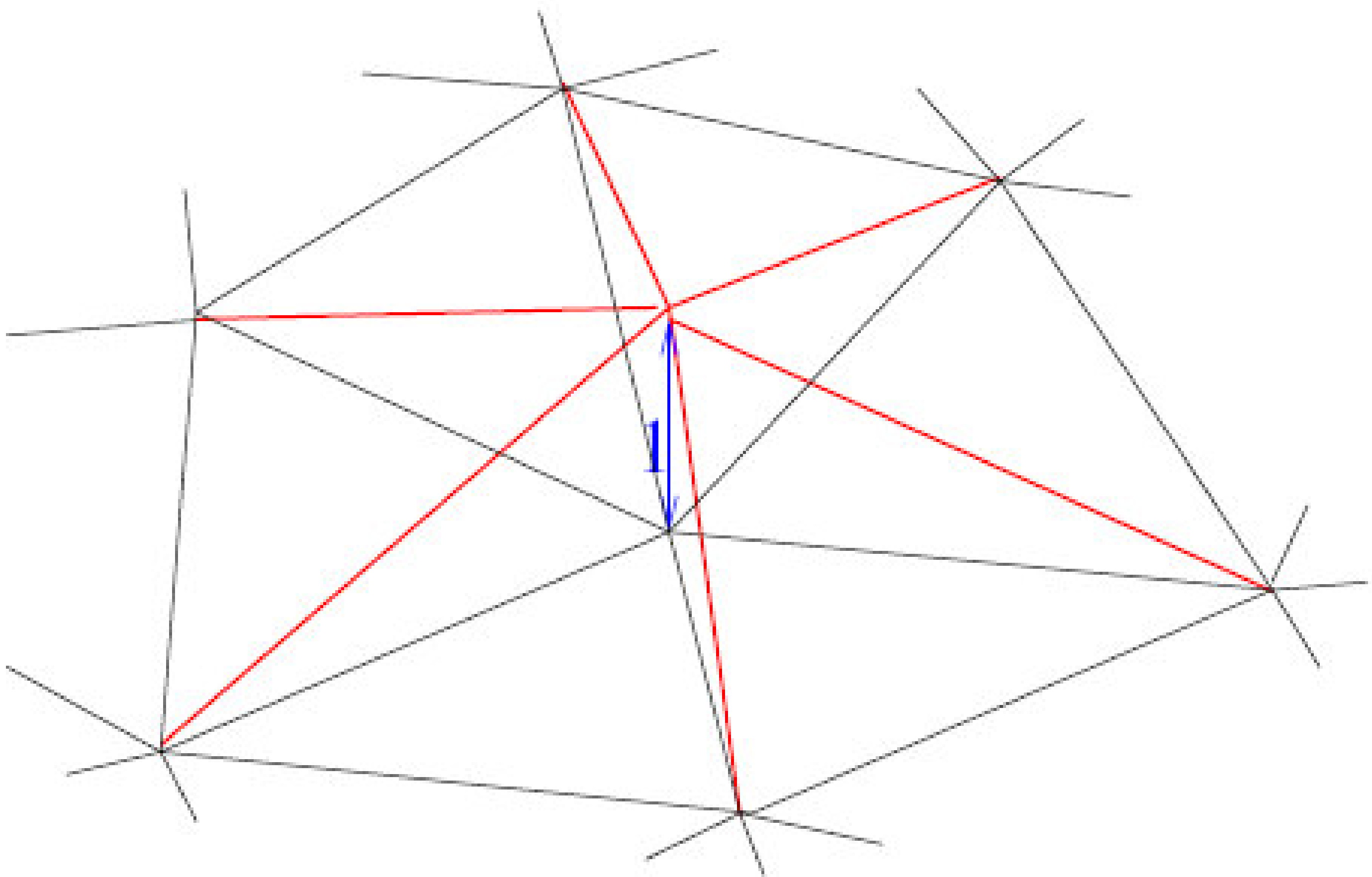


FEM nodal basis

Let $V_h(\mathcal{M}_h) = \mathcal{N}_h :=$ set of nodes of $\mathcal{M}_h$.
Then, the nodal basis is defined as: If $\mathcal{N}_h = \{a_1, \ldots, a_N\}$, nodal basis $\Phi_h := \{\phi_1, \ldots, \phi_N\}$ defined by $\phi_i(a_j) = \delta_{ij}$.

FEM nodal basis

Let $V_h(\mathcal{M}_h) = \mathcal{N}_h :=$ set of nodes of $\mathcal{M}_h$.
Then, the nodal basis is defined as: If $\mathcal{N}_h = \{a_1, \ldots, a_N\}$, nodal basis $\Phi_h := \{\phi_1, \ldots, \phi_N\}$ defined by $\phi_i(a_j) = \delta_{ij}$.



Finite element approximation

Now, we introduce the finite element approximation of the PML problem. From now on, we assume $g \in L^2(\Gamma_D)$. Let $b : H^1(\Omega_\rho) \times H^1(\Omega_\rho) \rightarrow \mathbb{C}$ be the sesquilinear form given by

$$b(\varphi, \psi) = \int_{\Omega_\rho} (A\nabla\varphi \cdot \nabla\bar{\psi} - \alpha\beta k^2 \varphi\bar{\psi}) dx.$$

Finite element approximation

Now, we introduce the finite element approximation of the PML problem. From now on, we assume $g \in L^2(\Gamma_D)$. Let $b : H^1(\Omega_\rho) \times H^1(\Omega_\rho) \to \mathbb{C}$ be the sesquilinear form given by

$$b(\varphi, \psi) = \int_{\Omega_\rho} (A\nabla\varphi \cdot \nabla\bar{\psi} - \alpha\beta k^2 \varphi\bar{\psi})dx.$$

Furthermore, denote by $H^1_{(0)}(\Omega_\rho) = \{v \in H^1(\Omega_\rho) : v = 0 \text{ on } \Gamma_\rho\}$. Then, we can write down the weak formulation for the PML problem: given $g \in L^2(\Gamma_D)$, find $\hat{u} \in H^1_{(0)}(\Omega_\rho)$ such that

$$b(\hat{u}, \psi) = \int_{\Gamma_D} g\bar{\psi}ds \qquad \forall \psi \in H^1_{(0)}(\Omega_\rho).$$

Finite element notation

- Let Γ_ρ^h, which consists of piecewise segments whose vertices lie on Γ_ρ, be an approximation of Γ_ρ.

Finite element notation

- Let Γ_ρ^h, which consists of piecewise segments whose vertices lie on Γ_ρ, be an approximation of Γ_ρ.
- Let $\mathcal{M}_h$ be a regular triangulation of the domain Ω_ρ^h.

Finite element notation

- Let Γ_ρ^h, which consists of piecewise segments whose vertices lie on Γ_ρ, be an approximation of Γ_ρ.
- Let $\mathcal{M}_h$ be a regular triangulation of the domain Ω_ρ^h.
- Assume the elements $K \in \mathcal{M}_h$ may have one curved edge align with Γ_D, such that $\Omega_\rho^h = \bigcup_{K\in\mathcal{M}_h} K$.

Finite element notation

- Let Γ_ρ^h, which consists of piecewise segments whose vertices lie on Γ_ρ, be an approximation of Γ_ρ.
- Let $\mathcal{M}_h$ be a regular triangulation of the domain Ω_ρ^h.
- Assume the elements $K \in \mathcal{M}_h$ may have one curved edge align with Γ_D, such that $\Omega_\rho^h = \bigcup_{K\in\mathcal{M}_h} K$.
- Let $V_h \subset H^1(\Omega_\rho^h)$ be the conforming linear finite element space over Ω_ρ^h, and $V_h^0 = \{v_h \in V_h : v_h = 0 \text{ on } \Gamma_\rho^h\}$.

Finite elements

Now, we can formulate the finite element approximation to the variational PML problem: find $u_h \in V_h^0$ such that

$$b(u_h, \psi_h) = \int_{\Gamma_D} g\bar{\psi}_h ds \qquad \forall \psi_h \in V_h^0.$$

and the discrete inf-sup condition

$$\sup_{0 \neq \psi_h \in V_h^0} \frac{|b(\varphi_h, \psi_h)|}{\|\psi_h\|_{H^1(\Omega_\rho)}} \geq \hat{\mu} \|\varphi_h\|_{H^1(\Omega_\rho)} \qquad \forall \varphi_h \in V_h^0, \hat{\mu} > 0.$$

Finite elements

Now, we can formulate the finite element approximation to the variational PML problem: find $u_h \in V_h^0$ such that

$$b(u_h, \psi_h) = \int_{\Gamma_D} g\bar{\psi}_h ds \qquad \forall \psi_h \in V_h^0.$$

and the discrete inf-sup condition

$$\sup_{0 \neq \psi_h \in V_h^0} \frac{|b(\varphi_h, \psi_h)|}{\|\psi_h\|_{H^1(\Omega_\rho)}} \geq \hat{\mu} \|\varphi_h\|_{H^1(\Omega_\rho)} \qquad \forall \varphi_h \in V_h^0, \hat{\mu} > 0.$$

Since we are interested in a posterior error estimates and the associated adaptive algorithm, we simply assume that the discrete problem has a unique solution $u_h \in V_h^0$.

Finite elements, definitions

- For any $K \in \mathcal{M}_h$, denote by h_K its diameter.

Finite elements, definitions

- For any $K \in \mathcal{M}_h$, denote by h_K its diameter.
- Let $\mathcal{B}_h$ denote the set of all sides that do not lie on Γ_D and Γ_ρ^h.

Finite elements, definitions

- For any $K \in \mathcal{M}_h$, denote by h_K its diameter.
- Let $\mathcal{B}_h$ denote the set of all sides that do not lie on Γ_D and Γ_ρ^h.
- For any $e \in \mathcal{B}_h, h_e$ stands for its length.

Finite elements, definitions

- For any $K \in \mathcal{M}_h$, denote by h_K its diameter.
- Let $\mathcal{B}_h$ denote the set of all sides that do not lie on Γ_D and Γ_ρ^h.
- For any $e \in \mathcal{B}_h, h_e$ stands for its length.
- For any $K \in \mathcal{M}_h$, introduce the residual $R_h := \nabla \cdot (A \nabla u_h|_K) + \alpha\beta k^2 u_h|_K$.

Finite elements, definitions

- For any $K \in \mathcal{M}_h$, denote by h_K its diameter.
- Let $\mathcal{B}_h$ denote the set of all sides that do not lie on Γ_D and Γ_ρ^h.
- For any $e \in \mathcal{B}_h, h_e$ stands for its length.
- For any $K \in \mathcal{M}_h$, introduce the residual
  $R_h := \nabla \cdot (A \nabla u_h|_K) + \alpha\beta k^2 u_h|_K$.
- For any interior side $e \in \mathcal{B}_h$, which is the common side of K_1 and $K_2 \in \mathcal{M}_h$, define the jump residual across e:
  $J_e := (A\nabla u_h|_{K_1} - A\nabla u_h|_{K_2}) \cdot \nu_e$, where the unit normal vector ν_e to e points from K_2 to K_1.

Finite elements, definitions

- For any $K \in \mathcal{M}_h$, denote by h_K its diameter.
- Let $\mathcal{B}_h$ denote the set of all sides that do not lie on Γ_D and Γ_ρ^h.
- For any $e \in \mathcal{B}_h, h_e$ stands for its length.
- For any $K \in \mathcal{M}_h$, introduce the residual $R_h := \nabla \cdot (A\nabla u_h|_K) + \alpha\beta k^2 u_h|_K$.
- For any interior side $e \in \mathcal{B}_h$, which is the common side of K_1 and $K_2 \in \mathcal{M}_h$, define the jump residual across e: $J_e := (A\nabla u_h|_{K_1} - A\nabla u_h|_{K_2}) \cdot \nu_e$, where the unit normal vector ν_e to e points from K_2 to K_1.
- If $e = \Gamma_D \cap \partial K$ for some element $K \in \mathcal{M}_h$, then we define the jump residual to be: $J_e := 2(\nabla u_h|_K \cdot \mathbf{n} + g)$.

Finite elements, definitions (cont)

- For any $K \in \mathcal{M}_h$, denote by η_K the local error estimator which is defined by

$$\eta_K = \max_{x \in \tilde{K}} w(x) \cdot \left(\|h_K R_h\|^2_{L^2(K)} + \frac{1}{2} \sum_{e \subset \partial K} h_e \|J_e\|^2_{L^2(e)} \right)^{1/2},$$

where $\tilde{K}$ is the union of all elements having nonempty intersection with K, and

$$w(x) = \begin{cases} 1 & \text{if } x \in \bar{\Omega}_R, \\ |\alpha_0 \alpha| e^{-k\Im(\tilde{r})\sqrt{1-\frac{r^2}{|\tilde{r}|^2}}} & \text{if } x \in \Omega^{\mathrm{PML}}. \end{cases}$$

Main Theorem

Theorem 3:
There exists a constant C depending only on the minimum angle of the mesh $\mathcal{M}_h$ such that the following a posterior error estimate is valid:

$$\begin{aligned}\|u-u_h\|_{H^1(\Omega_R)} &\le C\hat{C}^{-1}\sqrt{\Lambda(kR)}(1+kR)\left(\sum_{K\in\mathcal{M}_h}\eta_K^2\right)^{1/2}\\ &+ C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 e^{-k\Im(\tilde{\rho})\sqrt{1-\frac{R^2}{|\tilde{\rho}|^2}}}\|u_h\|_{H^{1/2}(\Gamma_R)},\end{aligned}$$

where $\Lambda(kR)=\max\left(1,\frac{|H_0^{(1)\prime}(kR)|}{|H_0^{(1)}(kR)|}\right)$.

Main Theorem

Theorem 3:
There exists a constant C depending only on the minimum angle of the mesh $\mathcal{M}_h$ such that the following a posterior error estimate is valid:

$$\begin{aligned}\|u-u_h\|_{H^1(\Omega_R)} &\leq C\hat{C}^{-1}\sqrt{\Lambda(kR)}(1+kR)\left(\sum_{K\in\mathcal{M}_h}\eta_K^2\right)^{1/2}\\ &+ C\hat{C}^{-1}(1+kR)^2|\alpha_0|^2 e^{-k\Im(\tilde{\rho})\sqrt{1-\frac{R^2}{|\tilde{\rho}|^2}}}\|u_h\|_{H^{1/2}(\Gamma_R)},\end{aligned}$$

where $\Lambda(kR)=\max\left(1,\frac{|H_0^{(1)\prime}(kR)|}{|H_0^{(1)}(kR)|}\right)$.

The important exponentially decaying factor $e^{-k\Im(\tilde{r})\sqrt{1-\frac{r^2}{|\tilde{r}|^2}}}$ in the PML region Ω^{PML} allows us to take thicker PML layers without introducing unnecessary fine meshes away from the fixed domain Ω_R.

Symmetry in $\hat{T}$

For any $\varphi \in H^1(\Omega_R)$, let $\tilde{\varphi}$ be its extension in Ω^{PML} such that

$$
\begin{aligned}
\nabla \cdot (\bar{A} \nabla \tilde{\varphi}) + \bar{\alpha}\bar{\beta} k^2 \tilde{\varphi} &= 0 \quad \text{in } \Omega^{\mathrm{PML}}, \\
\tilde{\varphi} &= \varphi \quad \text{on } \Gamma_R, \\
\tilde{\varphi} &= 0 \quad \text{on } \Gamma_\rho.
\end{aligned}
$$

Symmetry in $\hat{T}$

For any $\varphi \in H^1(\Omega_R)$, let $\tilde{\varphi}$ be its extension in Ω^{PML} such that

$$\begin{aligned} \nabla \cdot (\bar{A} \nabla \tilde{\varphi}) + \bar{\alpha}\bar{\beta} k^2 \tilde{\varphi} &= 0 \quad \text{in } \Omega^{\mathrm{PML}}, \\ \tilde{\varphi} &= \varphi \quad \text{on } \Gamma_R, \\ \tilde{\varphi} &= 0 \quad \text{on } \Gamma_\rho. \end{aligned}$$

Lemma 3:
Let (H2) be satisfied. For any $\varphi, \psi \in H^1(\Omega^{\mathrm{PML}})$, we have

$$\langle \hat{T}\varphi, \psi \rangle_{\Gamma_R} = \langle \hat{T}\bar{\psi}, \bar{\varphi} \rangle_{\Gamma_R}.$$

Symmetry in $\hat{T}$

For any $\varphi \in H^1(\Omega_R)$, let $\tilde{\varphi}$ be its extension in Ω^{PML} such that

$$\begin{aligned} \nabla \cdot (\bar{A}\nabla\tilde{\varphi}) + \bar{\alpha}\bar{\beta}k^2\tilde{\varphi} &= 0 \quad \text{in } \Omega^{\mathrm{PML}}, \\ \tilde{\varphi} &= \varphi \quad \text{on } \Gamma_R, \\ \tilde{\varphi} &= 0 \quad \text{on } \Gamma_\rho. \end{aligned}$$

Lemma 3:
Let (H2) be satisfied. For any $\varphi, \psi \in H^1(\Omega^{\mathrm{PML}})$, we have

$$\langle \hat{T}\varphi, \psi \rangle_{\Gamma_R} = \langle \hat{T}\bar{\psi}, \bar{\varphi} \rangle_{\Gamma_R}.$$

Whenever no confusion of the notation incurred, we shall write in the following $\tilde{\varphi}$ as φ in Ω^{PML}.

Error representation formula

Lemma 4:
For any $\varphi \in H^1(\Omega_R)$, which is extended to be a function in $H^1(\Omega_\rho)$, and $\varphi_h \in V_h^0$, we have

$$a(u - u_h, \varphi) = \int_{\Gamma_D} g(\overline{\varphi - \varphi_h}) - b(u_h, \varphi - \varphi_h) + \langle T u_h - \hat{T} u_h, \varphi \rangle_{\Gamma_R}.$$

Error representation formula

Lemma 4:

For any $\varphi \in H^1(\Omega_R)$, which is extended to be a function in $H^1(\Omega_\rho)$, and $\varphi_h \in V_h^0$, we have

$$a(u - u_h, \varphi) = \int_{\Gamma_D} g(\overline{\varphi - \varphi_h}) - b(u_h, \varphi - \varphi_h) + \langle Tu_h - \hat{T}u_h, \varphi \rangle_{\Gamma_R}.$$

Lets now prove this important Lemma!

Interpolation Operator

Since we are going to interpolate nonsmooth functions satisfying boundary conditions, we resort to an interpolation operator $\Pi_h : H^1_{(0)}(\Omega^h_\rho) \to V^0_h$ of Scott-Zhang.

Notation:

- Let $\mathcal{N}_h = \{a_i\}_{i=1}^N$ be the set of all nodes of $\mathcal{M}_h$.

Interpolation Operator

Since we are going to interpolate nonsmooth functions satisfying boundary conditions, we resort to an interpolation operator $\Pi_h : H^1_{(0)}(\Omega^h_\rho) \rightarrow V^0_h$ of Scott-Zhang.

Notation:

- Let $\mathcal{N}_h = \{a_i\}_{i=1}^N$ be the set of all nodes of $\mathcal{M}_h$.
- Let $\{\phi_i\}_{i=1}^N$ be the corresponding nodal basis of V_h.

Interpolation Operator

Since we are going to interpolate nonsmooth functions satisfying boundary conditions, we resort to an interpolation operator $\Pi_h : H^1_{(0)}(\Omega^h_\rho) \to V^0_h$ of Scott-Zhang.

Notation:

- Let $\mathcal{N}_h = \{a_i\}_{i=1}^N$ be the set of all nodes of $\mathcal{M}_h$.
- Let $\{\phi_i\}_{i=1}^N$ be the corresponding nodal basis of V_h.
- For any node a_i which is interior to Ω^h_ρ or on the boundary Γ_R, we take $\sigma_i = e$, any side in $\mathcal{B}_h$ having a_i as one of its vertex.

Interpolation Operator

Since we are going to interpolate nonsmooth functions satisfying boundary conditions, we resort to an interpolation operator $\Pi_h : H^1_{(0)}(\Omega^h_\rho) \to V^0_h$ of Scott-Zhang.

Notation:

- Let $\mathcal{N}_h = \{a_i\}_{i=1}^N$ be the set of all nodes of $\mathcal{M}_h$.
- Let $\{\phi_i\}_{i=1}^N$ be the corresponding nodal basis of V_h.
- For any node a_i which is interior to Ω^h_ρ or on the boundary Γ_R, we take $\sigma_i = e$, any side in $\mathcal{B}_h$ having a_i as one of its vertex.
- For any node a_i which is on the boundary Γ^h_ρ, we take σ_i as any side on Γ^h_ρ with one vertex a_i.

Interpolation Operator (cont)

- Let $a_{i,1} = a_i$, and $\{a_{i,j}\}_{j=1}^{2}$ the set of nodal points in σ_i with nodal basis $\{\phi_{i,j}\}_{j=1}^{2}$.

Interpolation Operator (cont)

- Let $a_{i,1} = a_i$, and $\{a_{i,j}\}_{j=1}^2$ the set of nodal points in σ_i with nodal basis $\{\phi_{i,j}\}_{j=1}^2$.
- Let $\{\psi_{i,j}\}_{j=1}^2$ be the $L^2(\sigma_i)$ dual basis:

$$\int_{\sigma_i} \psi_{i,j}(x)\phi_{i,k}(x)dx = \delta_{jk}, \qquad j,k = 1,2.$$

Interpolation Operator (cont)

We now define the interpolation operator $\Pi_h : H^1(\Omega_\rho^h) \to V_h$ to be

$$\Pi_h v(x) = \sum_{i=1}^{N} \phi_i(x) \int_{\sigma_i} \psi_i(x) v(x) dx.$$

One can show the following properties of Π_h:

- $\Pi_h v \in V_h^0$ if $v \in H_{(0)}^1(\Omega_\rho^h)$.
- $\|v - \Pi_h v\|_{L^2(K)} \le C h_k \|\nabla v\|_{L^2(\tilde{K})}$,
- $\|v - \Pi_h v\|_{L^2(e)} \le C h_e^{1/2} \|\nabla v\|_{L^2(\tilde{e})}$.

$\tilde{K}$ and $\tilde{e}$ denote the union of all elements in $\mathcal{M}_h$ having non-empty intersection with $K \in \mathcal{M}_h$ and the side e, respectively.

Fourth Part

IMPLEMENTATION AND EXAMPLES

Implementation

We use the a posteriori error estimate in the main theorem to determine the PML parameters. Just as before, we choose the PML medium property to be a power function. So, only the thickness $\rho - R$ of the layer and the medium parameter σ_0 are left to be specified.

Implementation

We use the a posteriori error estimate in the main theorem to determine the PML parameters. Just as before, we choose the PML medium property to be a power function. So, only the thickness $\rho - R$ of the layer and the medium parameter σ_0 are left to be specified.

First, we choose the exponentially decaying factor to be small such that it becomes negligible compared with the finite element discretization errors. Now, we set up an algorithm to adapt mesh size according to the a posteriori error estimate.

Algorithm

Let $TOL > 0$ be the tolerance for the error. Set $m = 2$. Now, the strategy is:

- Choose ρ and σ_0 such that the exponentially decaying factor $\hat{\omega} \leq 10^{-8}$;

Algorithm

Let $TOL > 0$ be the tolerance for the error. Set $m = 2$. Now, the strategy is:

- Choose ρ and σ_0 such that the exponentially decaying factor $\hat{\omega} \leq 10^{-8}$;
- Set the computational domain $\Omega_\rho = B_\rho \backslash \bar{\Gamma}_D$ and generate an initial mesh $\mathcal{M}_h$ over Ω_ρ;

Algorithm

Let $TOL > 0$ be the tolerance for the error. Set $m = 2$. Now, the strategy is:

- Choose ρ and σ_0 such that the exponentially decaying factor $\hat{\omega} \leq 10^{-8}$;
- Set the computational domain $\Omega_\rho = B_\rho \backslash \bar{\Gamma}_D$ and generate an initial mesh $\mathcal{M}_h$ over Ω_ρ;
- While ERR $>$ TOL do
  - refine the mesh $\mathcal{M}_h$: if $\eta_K > \frac{1}{2} \max_{\hat{K} \in \mathcal{M}_h} \eta_{\hat{K}}$, refine the element $K \in \mathcal{M}_h$;
  - solve the discrete problem (3.3) on $\mathcal{M}_h$;
  - compute error estimators on $\mathcal{M}_h$;

Algorithm

Let $TOL > 0$ be the tolerance for the error. Set $m = 2$. Now, the strategy is:

- Choose ρ and σ_0 such that the exponentially decaying factor $\hat{\omega} \leq 10^{-8}$;
- Set the computational domain $\Omega_\rho = B_\rho \backslash \bar{\Gamma}_D$ and generate an initial mesh $\mathcal{M}_h$ over Ω_ρ;
- While ERR $>$ TOL do
  - refine the mesh $\mathcal{M}_h$: if $\eta_K > \frac{1}{2} \max_{\hat{K} \in \mathcal{M}_h} \eta_{\hat{K}}$, refine the element $K \in \mathcal{M}_h$;
  - solve the discrete problem (3.3) on $\mathcal{M}_h$;
  - compute error estimators on $\mathcal{M}_h$;
- End While.

Example 1: Unit circle

Let the scatterer D be the unit circle. Let the exact solution be $u = H_0^{(1)}(kr)$, where $r = |x|$. Take $R = 2$, and $k = 1$. ($\rho = 4R$ and $\sigma_0 = 10$)

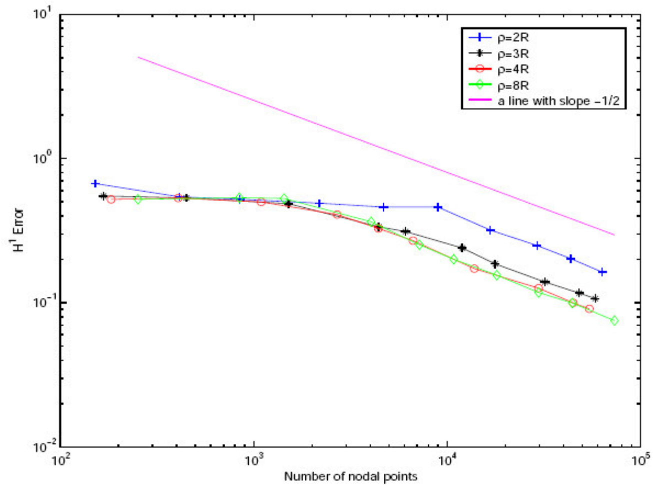


FIG. 5.2. *Quasi-optimality of the adaptive mesh refinements of the error* $\|\nabla(u-u_h)\|_{L^2(\Omega_R)}$ *for Example 1.*

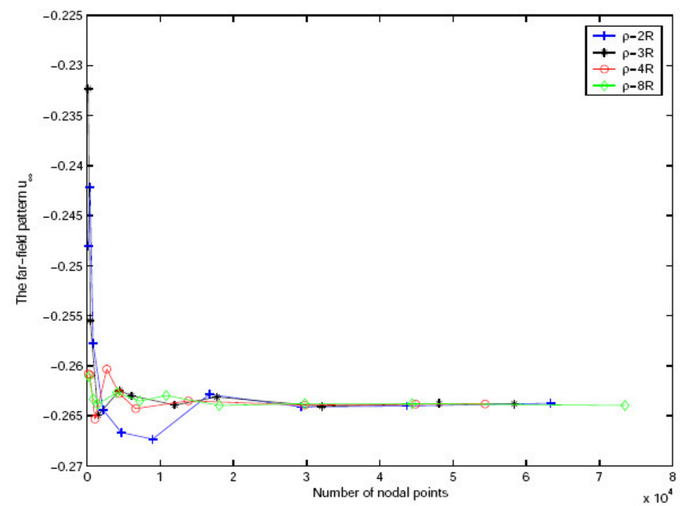


FIG. 5.3. *The real part of the far fields when the observing angle* $\theta = \pi/4$ *for Example 1.*

Example 1: Unit circle (cont)

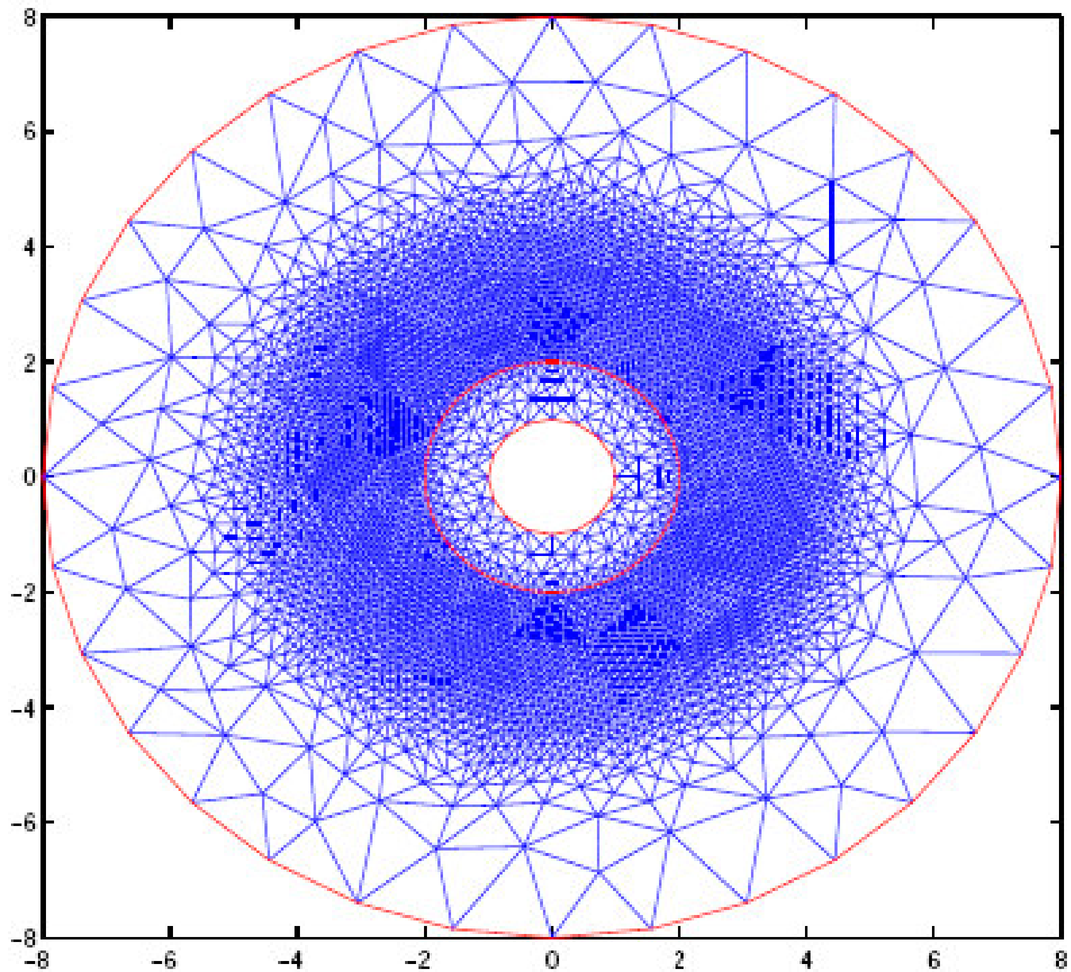


FIG. 5.4. *The mesh of 6668 nodes after 10 adaptive iterations when* $\rho = 4R$ *for Example 1.*

Example 2

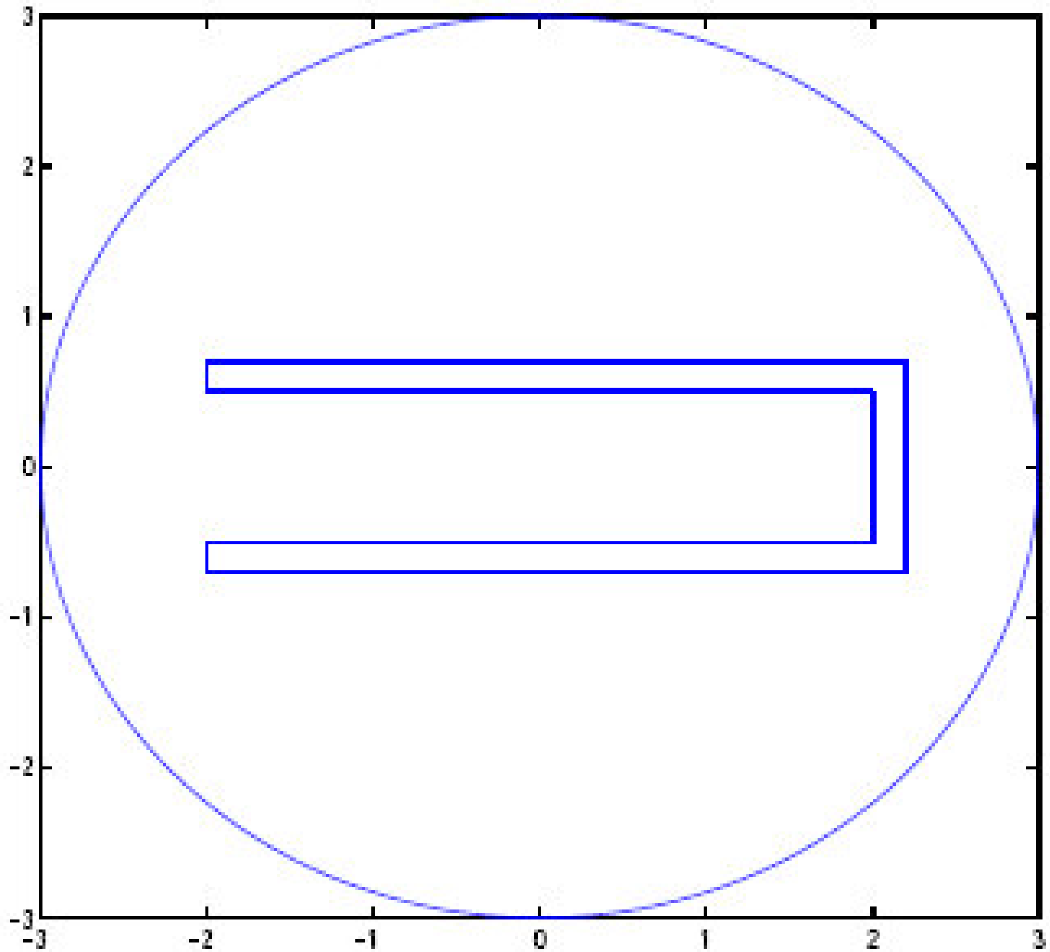


FIG. 5.1. *The geometry of the scatter for Example 2.*

Example 2 (cont)

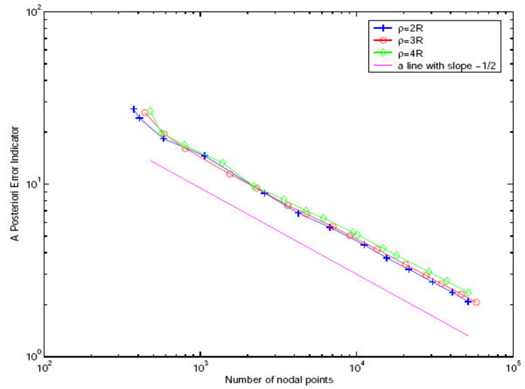


FIG. 5.5. *Quasi-optimality of the adaptive mesh refinements of the a posteriori error estimator for Example 2.*

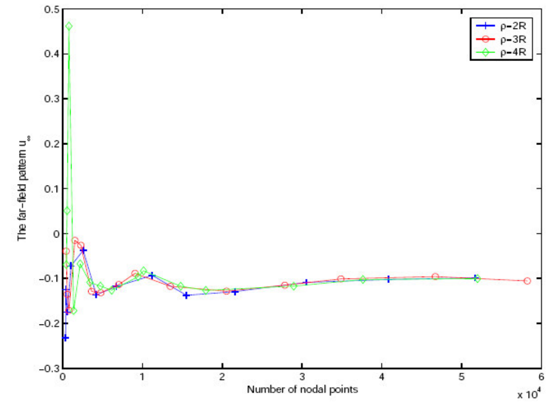


FIG. 5.6. *The real part of the far fields in the incident direction for Example 2.*

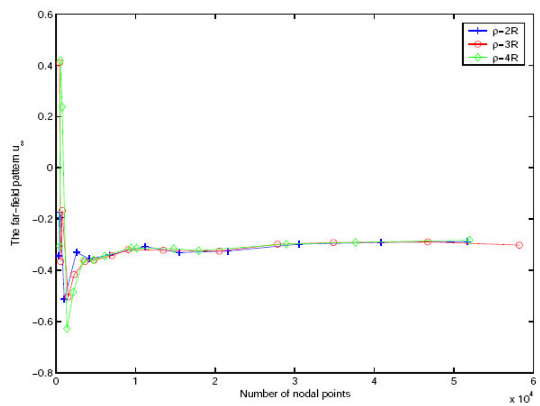


FIG. 5.7. *The real part of the far fields in the reflective direction for Example 2.*

Example 2

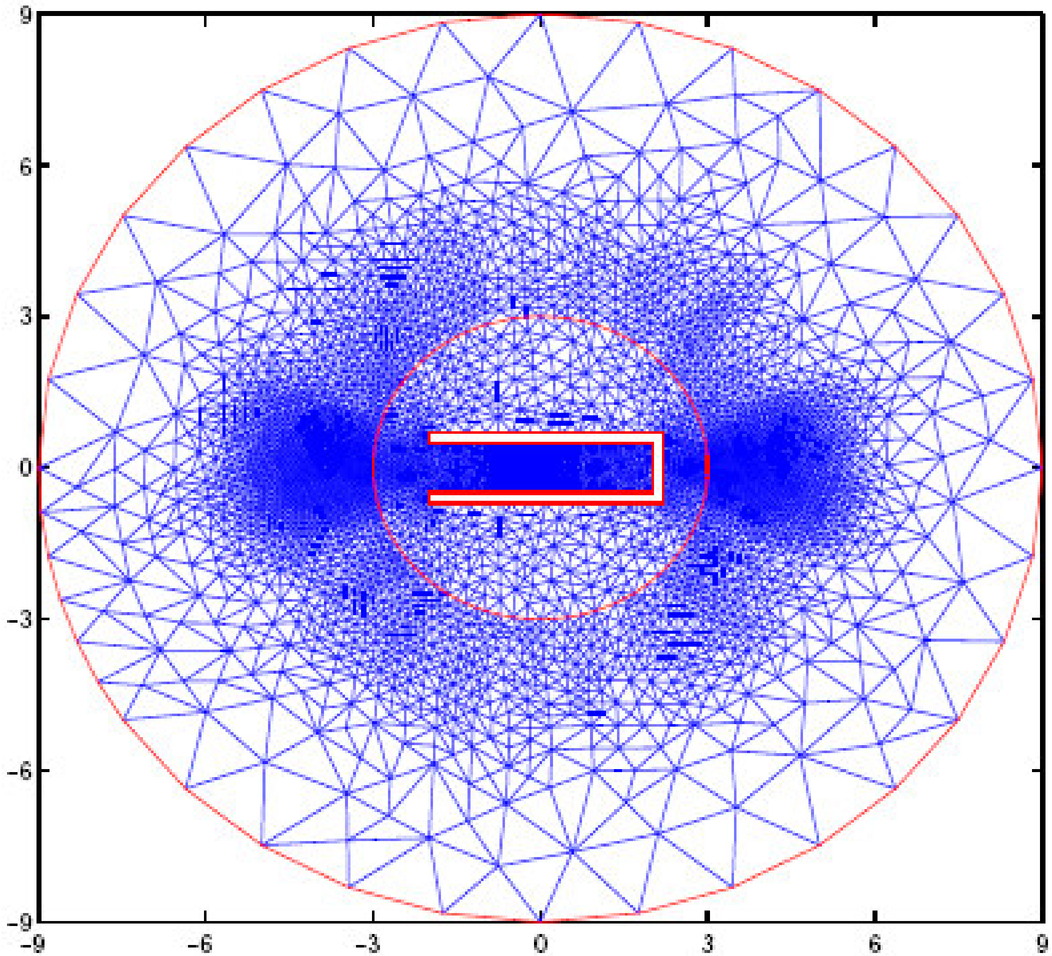


FIG. 5.8. *The mesh of 7048 nodes after 13 adaptive iterations when $\rho = 3R$ for Example 2.*

Example 2

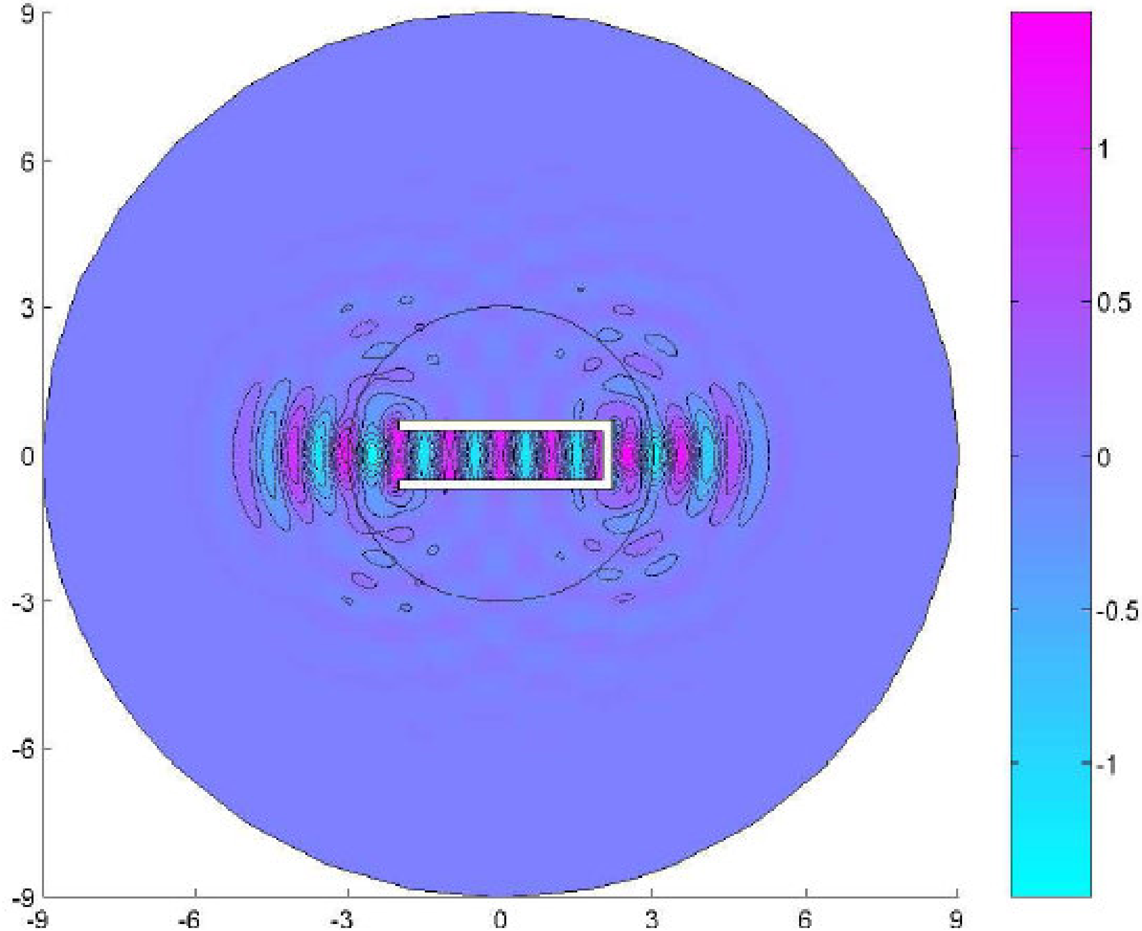


FIG. 5.9. *The contour plot of the real part of the solution when $\rho = 3R$ for Example 2.*

The End

Remarks / Questions