Perfectly Matched Layer Boundary Condition for Maxwell System
(using Finite Volume Time Domain Method)

2005 Applied Math Seminar
June, 14th 2005

Krishnaswamy **Sankaran**

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Laboratory for Electromagnetic Fields
and Microwave Electronics (IFH)

Outline of the talk

- Introduction to Maxwell system & FVTD
- Berenger's PML for Maxwell System
- Implementation issues
- Remarks & Conclusion

Maxwell System

- Maxwell system describes solution to two divergence and two curl equations of electric (E) and magnetic (H) field.

- In general for time domain analysis we concentrate on two maxwell curl equations describing space – time variation of these fields.

$$\nabla X \vec{H} - \varepsilon \frac{\partial \vec{E}}{\partial t} - \sigma \vec{E} = \vec{J}$$

$$\nabla X \vec{E} + \mu \frac{\partial \vec{H}}{\partial t} = \vec{K}$$

Maxwell System (continued...)

- For our analysis we consider only homogeneous form of Maxwell curl equations →

$$\sigma = 0 \quad .$$
$$\vec{J} = 0$$
$$\vec{K} = 0$$

$$\nabla X \vec{H} - \varepsilon \frac{\partial \vec{E}}{\partial t} = 0$$

$$\nabla X \vec{E} + \mu \frac{\partial \vec{H}}{\partial t} = 0$$

Maxwell System (continued...)

- Field quantities ***E*** and ***H*** are $\mathbb{R}^3$ vector-valued functions on space – time plane.
- Spatial domain is $\Omega \subset \mathbb{R}^3$ (possibly unbounded.)
- We consider finite time interval $\tau = (0, T) \subset \mathbb{R}_+$.
- Constitutive parameters: ε and μ are assumed to constant all over the domain.

Initial – Boundary Value problem

- The initial – boundary value problem we are interested in here is to find the functions ***E*** and ***H*** for $t \in \tau$ given that $\lim_{t \to 0} \vec{E}(x,t) = \lim_{t \to 0} \vec{H}(x,t) = 0 \quad \forall x \in \Omega$.

Maxwell system + Initial – boundary Conditions (perfect metallic, perfect magnetic, PML etc) → Initial – boundary value problem

- Above problem can be solved on computer taking into consideration of limited memory and time for processing.

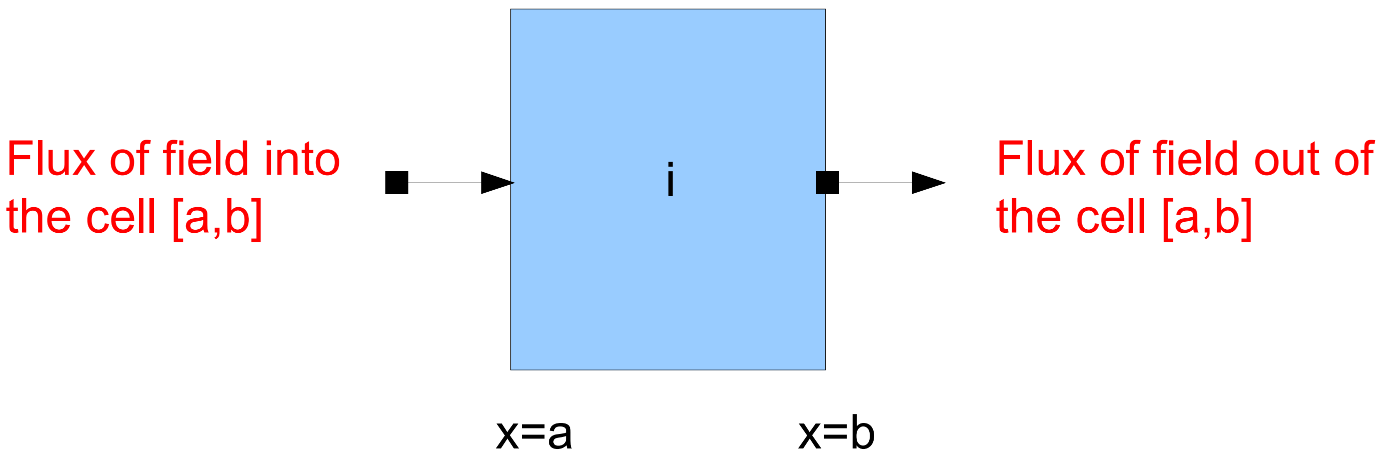
Introduction to FVTD Method

- FVTD stands for Finite Volume Time Domain

- Conceived from Computational Fluid Dynamics (CFD), FVTD works on conservation laws for any hyperbolic system.

- Basic idea is conservation of field quantities.

Finite Volume – Conservation Principle



- The time rate of change of the total field inside the section [a,b] changes only due to the flux of fields into and out of the pipe at the ends x=a and x=b.

Maxwell system in Conservative Form

$$Q_t + F_0(Q)_x + G_0(Q)_y = 0$$

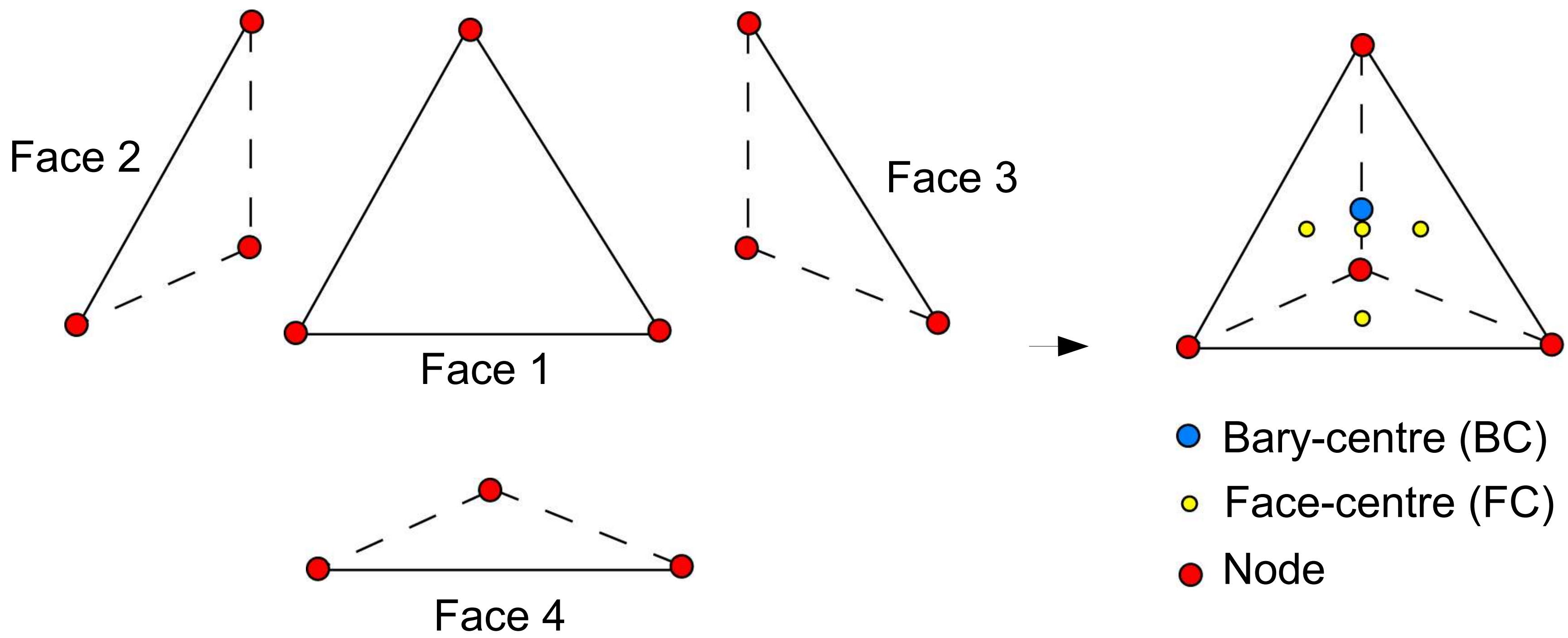
$$Q = (Q_1, Q_2, Q_3)^T = \begin{matrix} (\ H_x,\ H_y,\ E_z)^T & \text{TM case} \\ (-E_x, -E_y,\ H_z)^T & \text{TE case} \end{matrix}$$

$$F_0(Q) = (\ 0, -Q_3, -Q_2)^T$$
$$G_0(Q) = (\ Q_3,\ 0,\ Q_1)^T$$

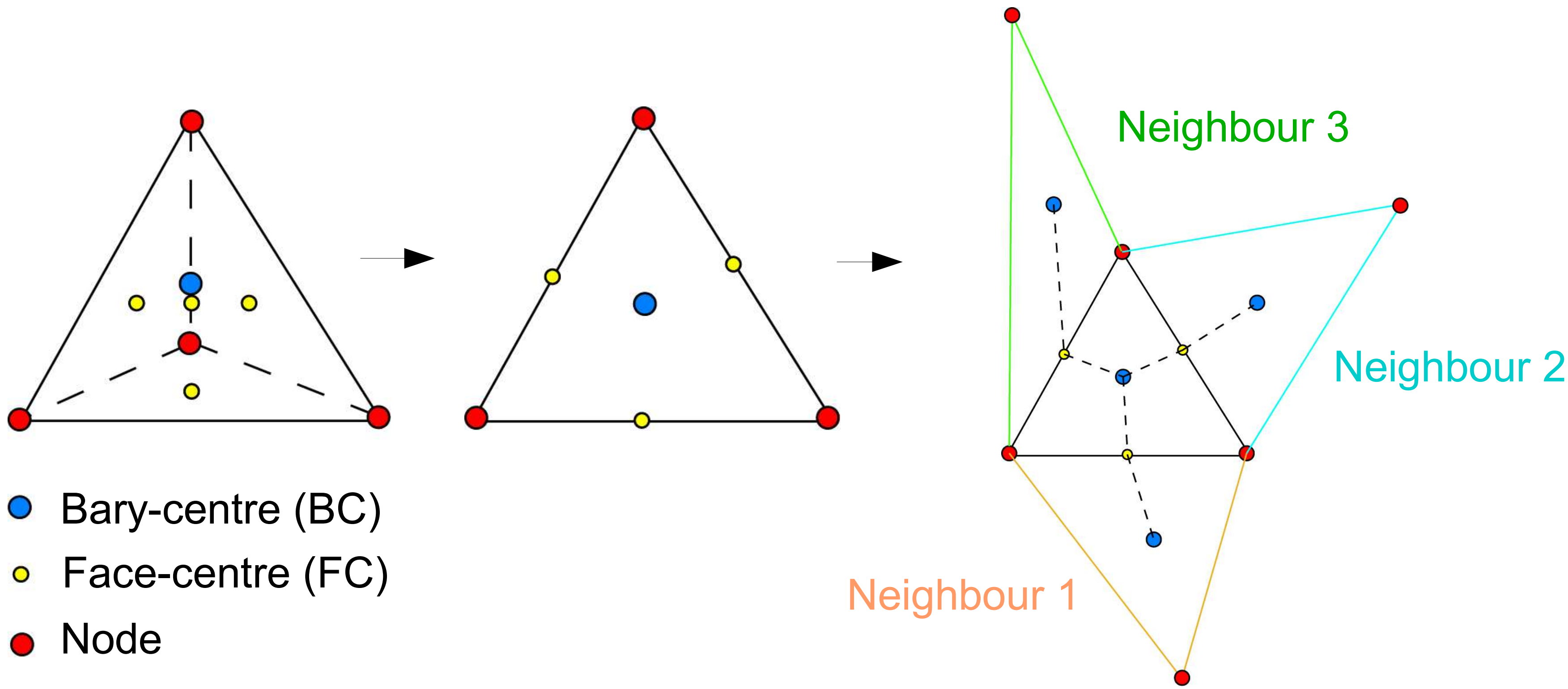
For our analysis we use only TM case

$$F_0(Q) = (\ 0, -E_z, -H_y)^T$$
$$G_0(Q) = (\ E_z,\ 0,\ H_x)^T$$

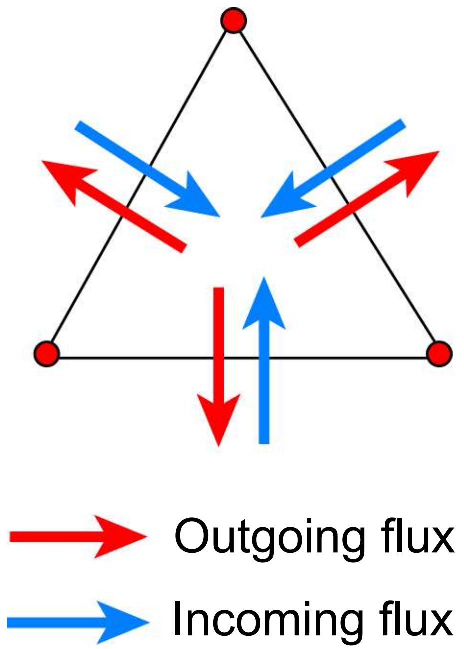
Finite Volumes in 3D

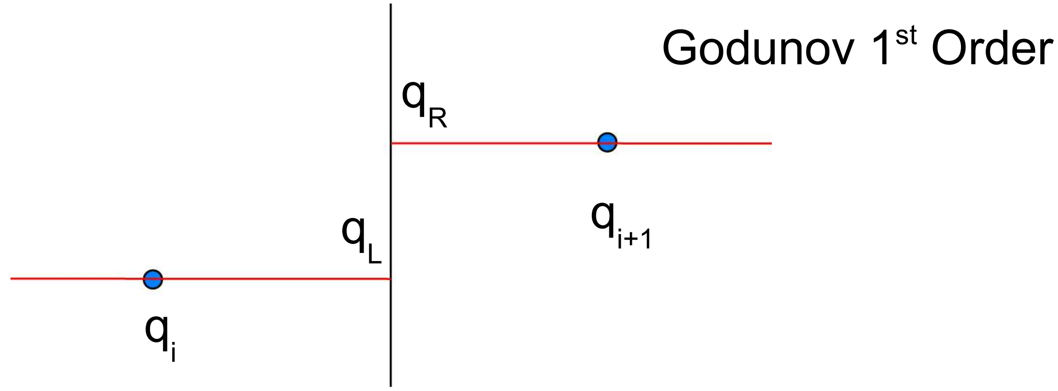


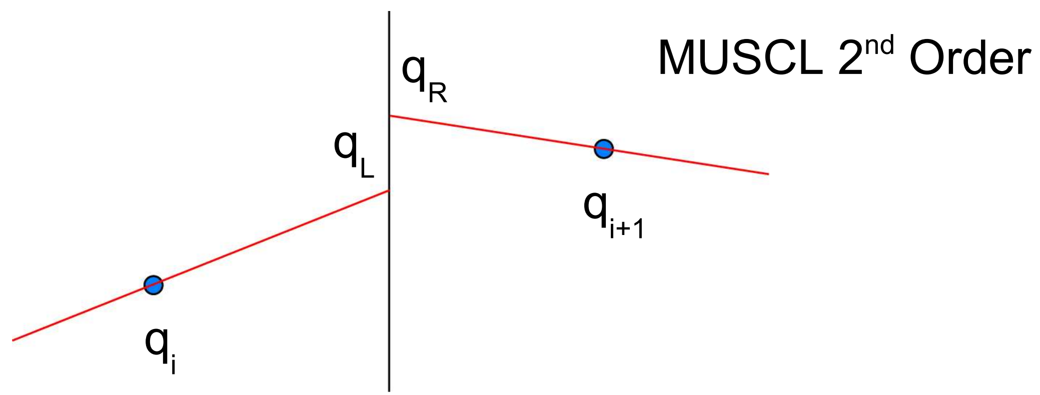
Finite Volumes in 2D



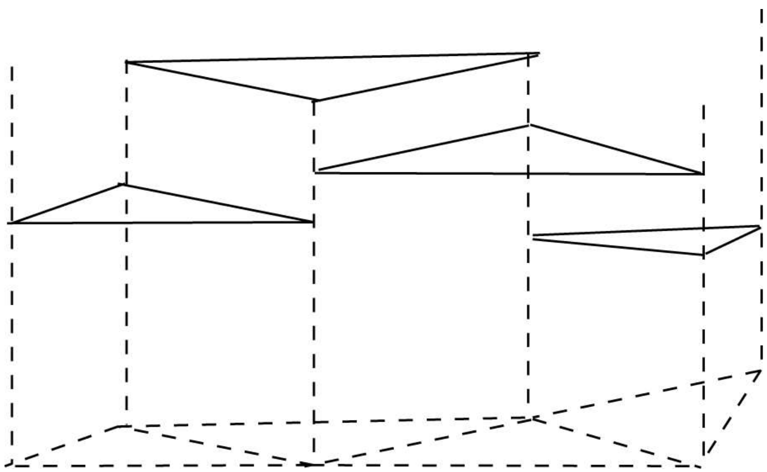
Edge Fluxes


Outgoing flux
Incoming flux

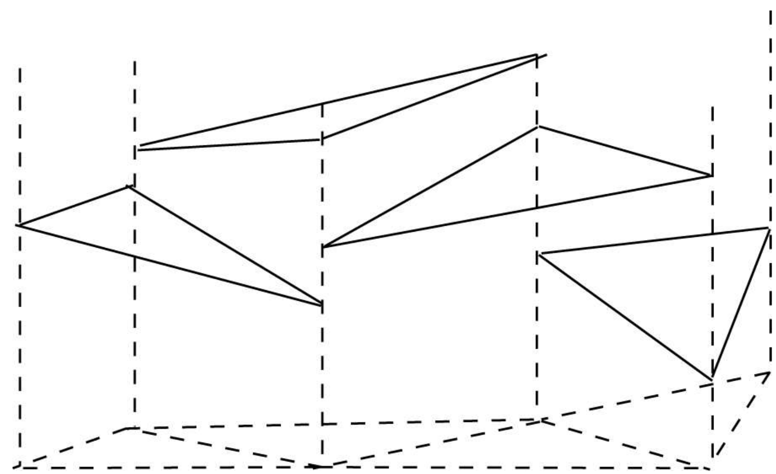

Godunov 1st Order
q_R
q_L
q_{i+1}
q_i


MUSCL 2nd Order
q_R
q_L
q_{i+1}
q_i

Flux approximation



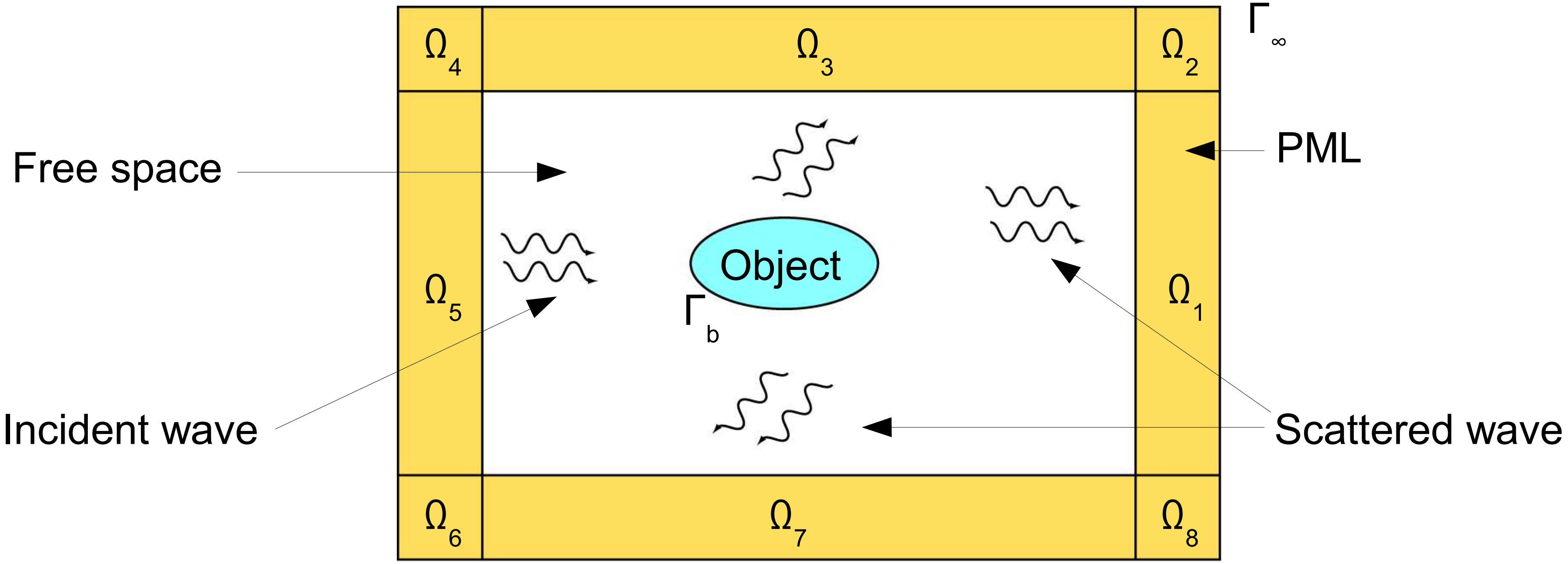
Piecewise constant
flux approximation



Piecewise linear
flux approximation

Berenger PML

- The method used in Berenger PML to absorb outgoing waves consists of limiting computational domain with an artificial boundary layer specially designed to absorb reflectionless the electromagnetic waves.



Berenger PML

- The computational domain is divided into two parts.

  Free space or vacuum – classical Maxwell equations.

  Absorbing Layer – modified Maxwell equations.

Modified Maxwell equation

$$\mu \frac{\partial \vec{H}}{\partial t} + \nabla X \vec{E} + \sigma_H \vec{H} = 0$$

$$\varepsilon \frac{\partial \vec{E}}{\partial t} - \nabla X \vec{H} + \sigma_E \vec{E} = 0$$

- σ_H and σ_E are magnetic and electric conductivities respectively.

Modified Maxwell system

- Modified Maxwell system can be considered as classical Maxwell system with source terms. To analyse the modified eqns at continuous levels leads to the condition: $\sigma_H = \sigma_E = \sigma$.

Modified Maxwell equation

$$\mu \frac{\partial \vec{H}}{\partial t} + \nabla X \vec{E} + \sigma \vec{H} = 0$$

$$\varepsilon \frac{\partial \vec{E}}{\partial t} - \nabla X \vec{H} + \sigma \vec{E} = 0$$

In FVTD formulation these terms are considered as source terms

$\sigma_H = \sigma_E = \sigma$ enables reflectionless transmission of a plane wave propagating normally across the interface between free space and outer boundary.

Berenger's PML

- J. P. Berenger published (J. Comp. Physics No. 114 – year 1994) this novel technique called PML in 2D case.

- With this new formulation, the theoretical reflection factor of a plane wave striking a vacuum – layer interface is zero at any incidence angle and at any frequency.

- We model this PML in 2D set-up . We make use of 2D Maxwell equations with TM formulation. Generalising to 3D full wave analysis is straightforward.

Berenger split field formulation

- We split $\mathbf{E}_z$ field into two subparts: $\mathbf{E}_{zx}$ and $\mathbf{E}_{zy}$. Hence we have four equations in modified Maxwell equations.

$$\mu \frac{\partial H_x}{\partial t} + \frac{\partial (E_{zx} + E_{zy})}{\partial y} + \sigma_y H_x = 0$$

$$\mu \frac{\partial H_y}{\partial t} - \frac{\partial (E_{zx} + E_{zy})}{\partial x} + \sigma_x H_y = 0$$

$$\varepsilon \frac{\partial E_{zx}}{\partial t} - \frac{\partial H_y}{\partial x} + \sigma_x E_{zx} = 0$$

$$\varepsilon \frac{\partial E_{zy}}{\partial t} + \frac{\partial H_x}{\partial y} + \sigma_y E_{zy} = 0$$

- Magnetic and electric conductivities are also split into σ_{Hx}, σ_{Hy}, σ_{Ex} and σ_{Ey} with conditions $\sigma_{Hx} = \sigma_{Ex} = \sigma_x$ and $\sigma_{Hy} = \sigma_{Ey} = \sigma_y$.

σ_x and σ_y – Physical Interpretation

- Choice of σ_x and σ_y is very critical to obtain perfectly transparent vacuum - layer interfaces for outgoing waves.
- σ_x can be interpreted as absorption coefficient along *x*-direction. Correspondingly σ_y is along *y*-direction.

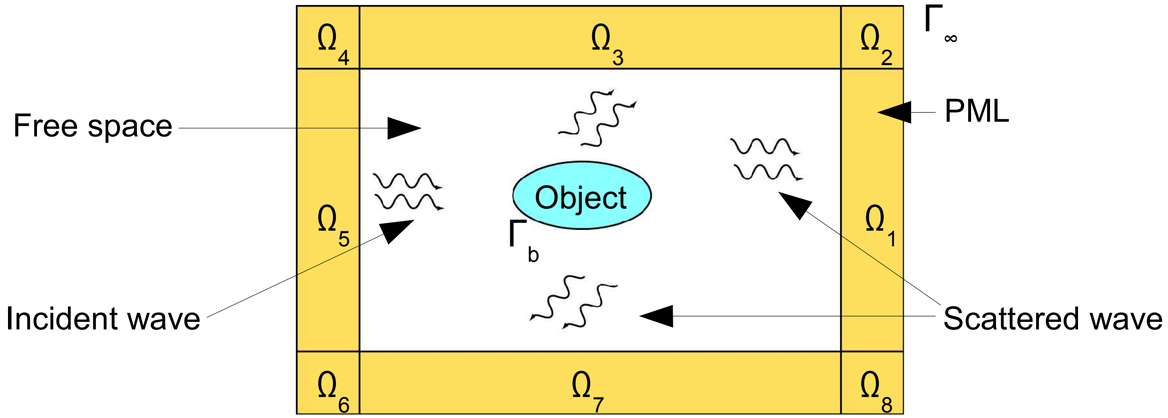
$$\textit{If } \vec{e}_x \textit{ is the normal direction for the interface between}$$
$$\textit{free space} - \textit{PML medium then}$$
$$\gamma = 0 \quad \forall \theta_i \text{ and } \forall \nu \quad \textit{if } \sigma_y = 0$$
$$\gamma = \textit{reflection coefficient} \quad \theta_i = \textit{incidence angle}$$
$$\nu = \textit{wave frequency}$$

$$\textit{Similarly if } \vec{e}_y \textit{ is the normal direction for the interface between}$$
$$\textit{free space} - \textit{PML medium then}$$
$$\gamma = 0 \quad \forall \theta_i \text{ and } \forall \nu \quad \textit{if } \sigma_x = 0$$

Conductivity choices

- Computational domain is bounded in all sides by artificial absorbing layers namely Ω_1 to Ω_8.

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_8 \quad where \quad \begin{aligned} \Omega_1 &= (x,y);\ y \in [-b,b],\ x \in [\ a, A] \\ \Omega_2 &= (x,y);\ y \in [\ b,B],\ x \in [\ a, A] \\ \Omega_3 &= (x,y);\ y \in [\ b,B],\ x \in [-a, a] \end{aligned}$$



- Also to avoid parasitic reflections on the interface of the free space and PML medium, we take $\sigma_y = 0$ in Ω_1 and $\sigma_x = 0$ in Ω_3 etc.

Conductivity choices (continued...)

- Based on the discussions before we can more precisely define conductivity choices in different portions of artificial boundary.

$$
\begin{aligned}
\vec{\sigma} &= \sigma_x \vec{e}_x + \sigma_y \vec{e}_y \\
\vec{\sigma}_1 &= \sigma_0 \left(\frac{x-a}{A-a} \right)^n \vec{e}_x \\
\vec{\sigma}_3 &= \sigma_0 \left(\frac{y-b}{B-b} \right)^n \vec{e}_y \\
\vec{\sigma} &= \vec{\sigma}_1 \ \text{in} \ \Omega_1 \\
\vec{\sigma} &= \vec{\sigma}_3 \ \text{in} \ \Omega_3 \\
\vec{\sigma} &= \vec{\sigma}_1 + \vec{\sigma}_3 \ \text{in} \ \Omega_2
\end{aligned}
$$

- Choice of $\boldsymbol{\sigma}_0$ and **n** play a vital role in formulating reflectionless boundary condition. Different possibilities are disscussed here.

Conductivity choices (continued...)

- One another possible choice of σ_0 can be done as presented paper of F. Collino, P.B. Monk (Comput. Methods Appl. Mech. Engrg. No. 164 year 1998 pg 157 – 171.)

$$\delta = \frac{2\pi c}{\omega} \qquad (\textit{layer length} = 1\,\textit{wavelength})$$

$$\vec{\sigma}(x) = \sigma_0 \left(\frac{x-a}{\delta}\right)^2 \vec{e}_x, \quad \forall\, x > a \qquad (\textit{parabolic}-\textit{law})$$

$$\vec{\sigma}(y) = \sigma_0 \left(\frac{y-a}{\delta}\right)^2 \vec{e}_y, \quad \forall\, y > b$$

$$\vec{\sigma}_0 = \frac{3}{2\delta}\log_e\left(R_0^{-1}\right) \qquad R_0 = 10^{-2},\ 10^{-3},\ 10^{-4}$$

Implementation issues

- A few implementation issues concerning PML formulation are to be discussed in depth before actual coding procedure.

Flux calculation in PML layer leads to solving a non-hyperbolic equation – New formulation of Maxwell eqns.

Issues on hyperbolicity of new formulation

Termination of PML using PMC or ABC boundary condition

PMC – Perfect Magnetic Conducting boundary condition
ABC – Absorbing Boundary Condition

Loss of hyperbolicity of the system

- The modified Maxwell equations are not purely hyperbolic.

$$
\begin{aligned}
\mu \frac{\partial H_x}{\partial t} + \frac{\partial (E_{zx} + E_{zy})}{\partial y} + \sigma_y H_x &= 0 \\
\mu \frac{\partial H_y}{\partial t} - \frac{\partial (E_{zx} + E_{zy})}{\partial x} + \sigma_x H_y &= 0 \\
\varepsilon \frac{\partial E_{zx}}{\partial t} - \frac{\partial H_y}{\partial x} + \sigma_x E_{zx} &= 0 \\
\varepsilon \frac{\partial E_{zy}}{\partial t} + \frac{\partial H_x}{\partial y} + \sigma_y E_{zy} &= 0
\end{aligned}
$$

- The splitting of $\mathbf{E_z}$ field into $\mathbf{E_{zx}}$ and $\mathbf{E_{zy}}$ fields spoils the hyperbolic nature of the system and hence we need to manipulate the above equations to solve them numerically .

Implementation issues

- For numerical simplicity, we can choose to conserve the field components in vacuum (**H**$_x$, **H**$_y$, **E**$_z$.) Hence if we can change **E**$_{zx}$ by **E**$_z$ - **E**$_{zy}$ we can formulate a set of four modified Maxwell equations which are more easier to handle and analyse.

$$\mu \frac{\partial H_x}{\partial t} + \frac{\partial E_z}{\partial y} + \sigma_y H_x = 0$$

$$\mu \frac{\partial H_y}{\partial t} - \frac{\partial E_z}{\partial x} + \sigma_x H_y = 0$$

$$\varepsilon \frac{\partial E_z}{\partial t} + \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} + \sigma_x E_z + (\sigma_y - \sigma_x) E_{zy} = 0$$

$$\varepsilon \frac{\partial E_{zy}}{\partial t} + \frac{\partial H_x}{\partial y} + \sigma_y E_{zy} = 0$$

Classical Maxwell Eqns with source terms

Still this is an non-hyperbolic Eqn.

PML – Is it well-posed???

- The Jacobian matrix ***A*** contains valuable information regarding the flux function and could be used to study eigenvalues and eigen -vectors of the system. The previous set of modified Maxwell eqns can be written in condensed form.

$$Q_t + \vec{\nabla} \boldsymbol{F}(Q) + \sum(Q) = 0 \qquad \text{where } \boldsymbol{F}(Q) = (F(Q), G(Q))^T$$

$$Jacobian\, A = A(\vec{n}) = \vec{n}\, \boldsymbol{F}'(Q) = n_1 \frac{\partial F}{\partial Q}(Q) + n_2 \frac{\partial G}{\partial Q}(Q)$$

- Jacobian ***A*** has three real eigenvalues – with a double mulplicity of zero (Jordan block of dimension 2.) This makes the resulting system non - hyperbolic.

PML – Is it well-posed??? (continued...)

- But it has been proved by de la Bourdonnaye that if we add the divergence and an additional compatibility conditions the resulting system has the property of well-posedness as a hyperbolic system.

$$\text{Compatibility Eqn}: \quad \Delta E_{zy} = \frac{\partial^2}{\partial y^2} E_z$$

- It is also worth to note that this equation is redundant for initial data verifying these constraints because $\partial_t(\Delta E_{zy}) = \partial_t\left(\partial^2/\partial y^2 E_z\right)$.

- We also impose at $t = 0$, in the PML $E_z = E_{zy} = 0$.

- Hence the PML formulation is well – posed !!!.

PML flux approximation

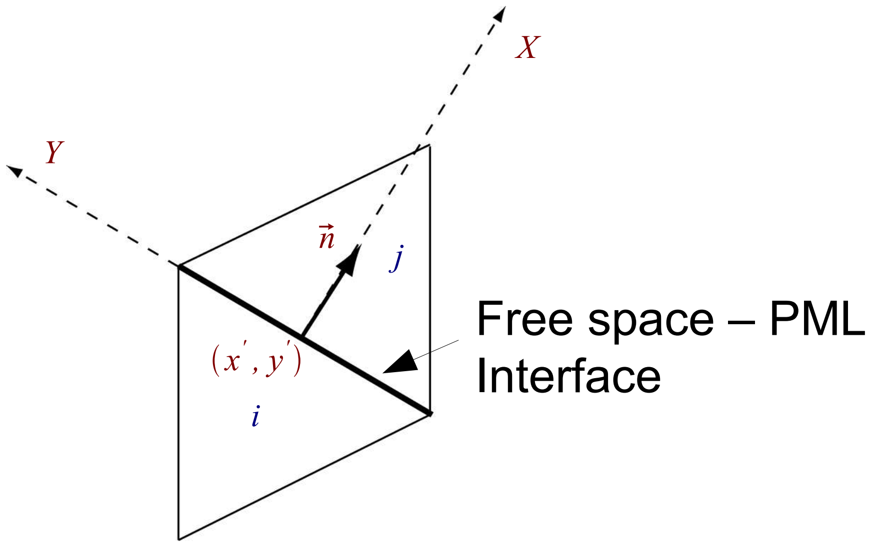
- First three equations (out of four) : classical Maxwell system with source terms.
- Our attention is to approximate the flux φ for the fourth equation.
- φ is totally determined by our knowledge of $\boldsymbol{H}_{x'}$.
- We can solve for $\boldsymbol{H}_{x'}$ by solving a Riemann problem at the interface between two neighbour cells.

$$Q_t + F(Q)_x + G(Q)_y = 0 \quad \rightarrow \quad \textit{Bidimensional Riemann problem!}$$

$$Q(x, y, 0) = \begin{cases} H_x(i) & \text{if } n_1 x + n_2 y < n_1 x' + n_2 y' \\ H_x(j) & \text{if } n_1 x + n_2 y > n_1 x' + n_2 y' \end{cases}$$

PML flux approximation (continued...)

- For FVTD in a triangular mesh this is determined based on some thumb-rules .



$$\begin{aligned} \vec{n} &\rightarrow \textit{normal vector} \\ (x', y') &\rightarrow \textit{edge centre coordinates} \\ i &\rightarrow \textit{neighbour } 1 \\ j &\rightarrow \textit{neighbour} 2 \\ X &\rightarrow X-\textit{direction} \\ Y &\rightarrow Y-\textit{direction} \end{aligned}$$

- But the field $\boldsymbol{H_{x'}}$ is invariant along Y-direction.

$$Q_t + F(Q)_x = 0 \rightarrow \textit{Monodimensional Riemann problem!}$$

$$Q(x,0) = \begin{cases} H_x(i) & \text{if } X < 0 \\ H_x(j) & \text{if } X > 0 \end{cases}$$

PML flux approximation (continued...)

- Using the Rankine – Hugoniot jump relation, we can formulate the value of $\boldsymbol{H}_x$ and $\boldsymbol{H}_y$ in each neighbours of each interfaces.
- For TM case the PML flux function can be obtained with only the knowledge of $\boldsymbol{H}_x$ and $\boldsymbol{E}_z$ in each neighbours of each interfaces.

$$\phi_{pml} = \boldsymbol{f}\left(H_x(i),\ H_x(j),\ E_z(i),\ E_z(j),\ n_2\right)$$

$$\phi_{pml} = \frac{1}{2}\left(H_x(i) + H_x(j)\right)n_2 - \frac{1}{2}\left(E_z(i) + E_z(j)\right)n_2^2$$

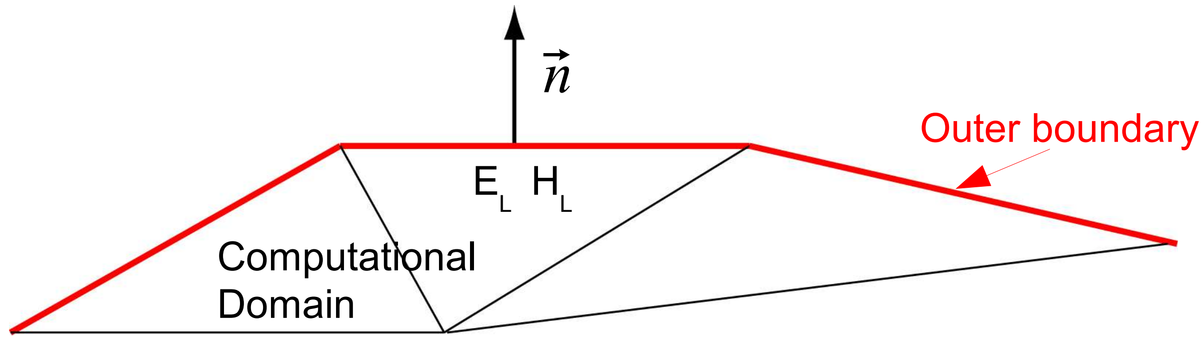
Upwind flux

Correction factor

Treatment of outer boundary condtions

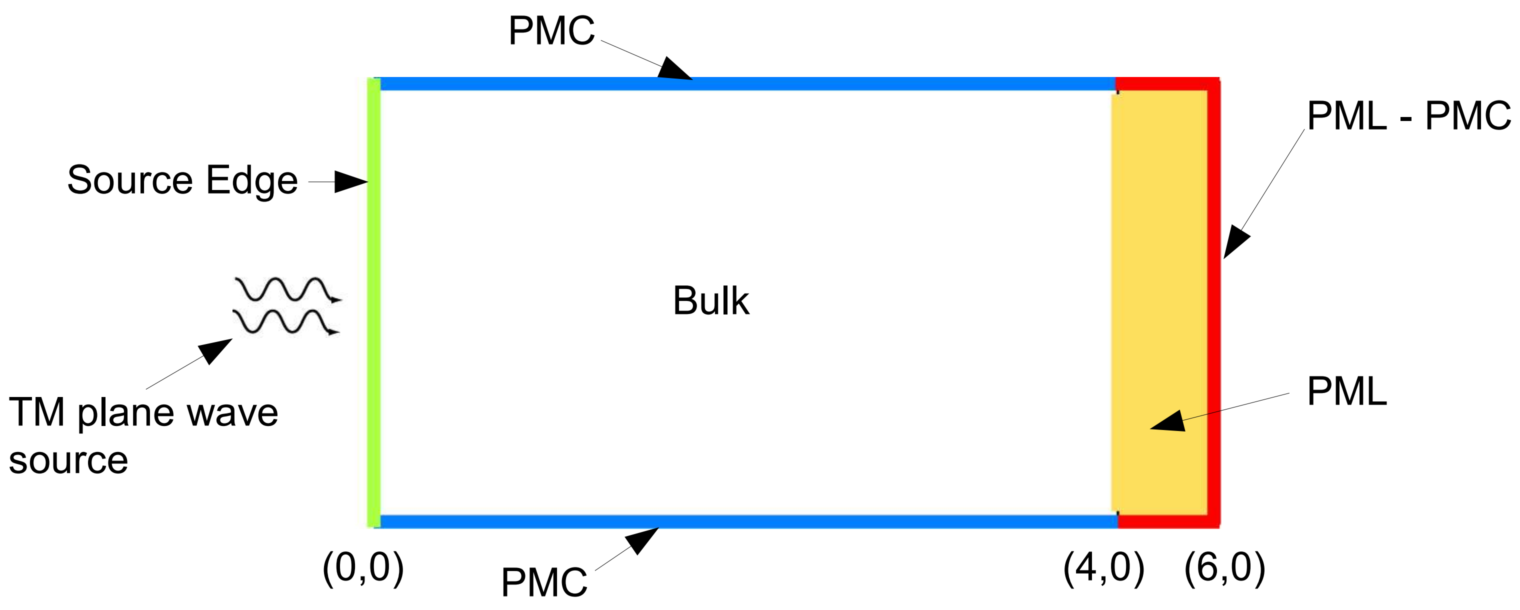
- Different chooses for outer boundary conditions are possible to terminate the PML.
- **PEC** – Perfect Electric Conductor : $\vec{n} \ X \ \vec{E} = 0$
- **PMC** – Perfect Magnetic Conductor : $\vec{n} \ X \ \vec{H} = 0$
- **SM-ABC** – Silver – Mueller Absorbing Boundary Condition:

$$\sqrt{\frac{\epsilon_0}{\mu_0}} \ \vec{n} \ X \ \vec{E}_L + \vec{n} \ X \ (\vec{n} \ X \ \vec{H}_L) = 0$$



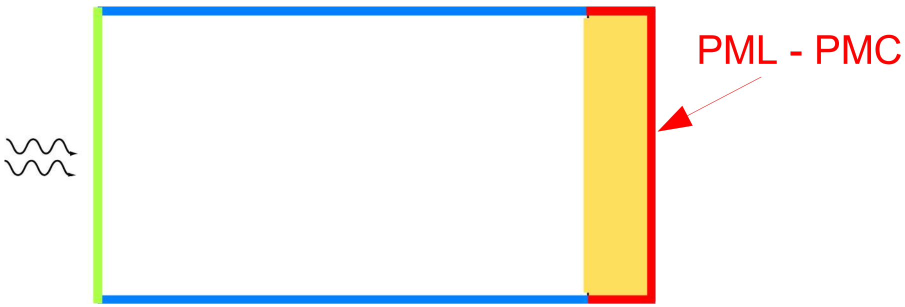
Experiments Done !!!

- A first – order (in space and time discretisation) scheme was successfully tested for the presented work and numerical results are shown here.
- For the sake of fast and robust code validation a simplified PML setup was chosen for simulation.
- Computational domain used:



Experiments Done !!! (continued...)

- A few words on PML – PMC flux function is mandatory to complete the description of the simulation setup.



- For a TM formulation the flux function for PML – PMC is given by:

$$\int_{\partial C_i \cap \Gamma_\infty} \boldsymbol{F}(Q)\vec{n}\; d\sigma = \begin{pmatrix} n_2 \; E_{zL} \\ -n_1 \; E_{zL} \\ 0 \\ n_2 H_{xL} \end{pmatrix}$$

Remarks & Conclusions

- The presented FVTD based PML was successfully implemented and tested at different spatial discretisations.

- The convergence of the result is clearly observed when reducing spatial and temporal discretisation.

- Many minute details regarding the PML were tried and some interesting conclusions regarding PML thickness were analysed. The choice of $\boldsymbol{\sigma_0}$ and **n** were found to very critical for very good PML formulation.

- Last but not least, it was a nice experience to model the basic finite difference model of Berenger's PML in FVTD unstructured formulation. This gave a deeper insight into the scheme and also about PML.

Thanks & Acknowledgements

- I am greatly indebted and thankful to Dr. Sebastien Tordeux for his encouragement in finishing the implementation of PML for FVTD. Without his valuable suggestions it would have been impossible to finish it in this shape.

- I am grateful to Prof. Dr. Ralf Hiptmair who highly encouraged me and supported my suggestions for doing FVTD based PML. Thanks for this wonderful opportunity.

Questions & Comments !!!

Questions ???
Comments !!!